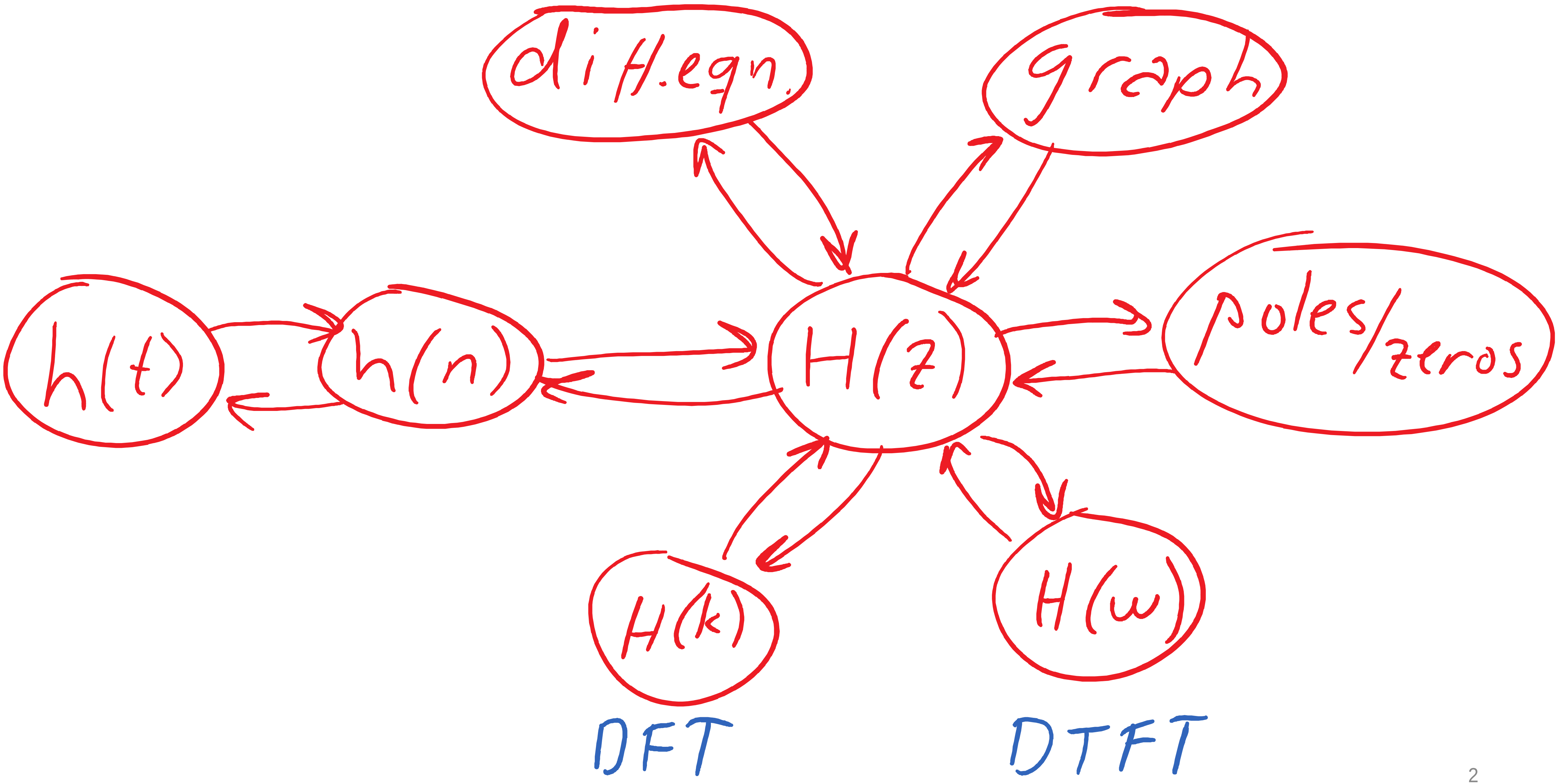
Lecture 10

Digital Signal Processing

Chapter 7

Discrete Fourier transform
DFT


diff.eqn.
graph
h(t)
h(n)
H(z)
poles/zeros
H(k)
H(ω)
DFT
DTFT

We have previously defined the DTFT

The Discrete-Time Fourier Transform (DTFT)

The Fourier transform of a discrete signal:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\mathrm{e}^{-\mathrm{j}\omega n}$$

$$x(n) = \int_{-\pi}^{\pi} X(\omega)\mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\overset{\omega}{\cancel{t}} = \int_{0}^{2\pi} X(\omega)\mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\overset{\omega}{\cancel{t}}$$

periodic with period
$w = 2\pi$

We have previously defined the Z-transform

The z-transform

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

if h(n) causal

$$h(n) = 0 \text{ for } n < 0$$

If $h(n)$ is causal and stable then

$$H(\omega) = H(z \,|\, z = e^{j\omega})$$

The Discrete Fourier Transform (DFT)

Def:

DFT

$$X_{\mathrm{DFT}}(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi \cdot \frac{k}{N} \cdot n} \qquad \text{for } k = 0, 1, \ldots, N-1$$

Inverse DFT

$$x_{\mathrm{DFT}}(n) = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k) e^{j2\pi \cdot \frac{n}{N} \cdot k} \qquad \text{for } \cancel{k}\ n = 0, 1, \ldots, N-1$$

Compare the DTFT and the DFT

DTFT: $$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

If $x(n)$ is causal and has length N we get sampling of ω

$$\omega = \underbrace{2\pi \cdot \left\{ 0 \quad \frac{1}{N} \quad \frac{2}{N} \quad \frac{3}{N} \quad \cdots \quad \frac{N-1}{N} \right\}}$$

DFT: $$X_{\mathrm{DFT}}(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi \cdot \frac{k}{N} \cdot n} \qquad \text{for } k = 0, 1, \ldots, N-1$$

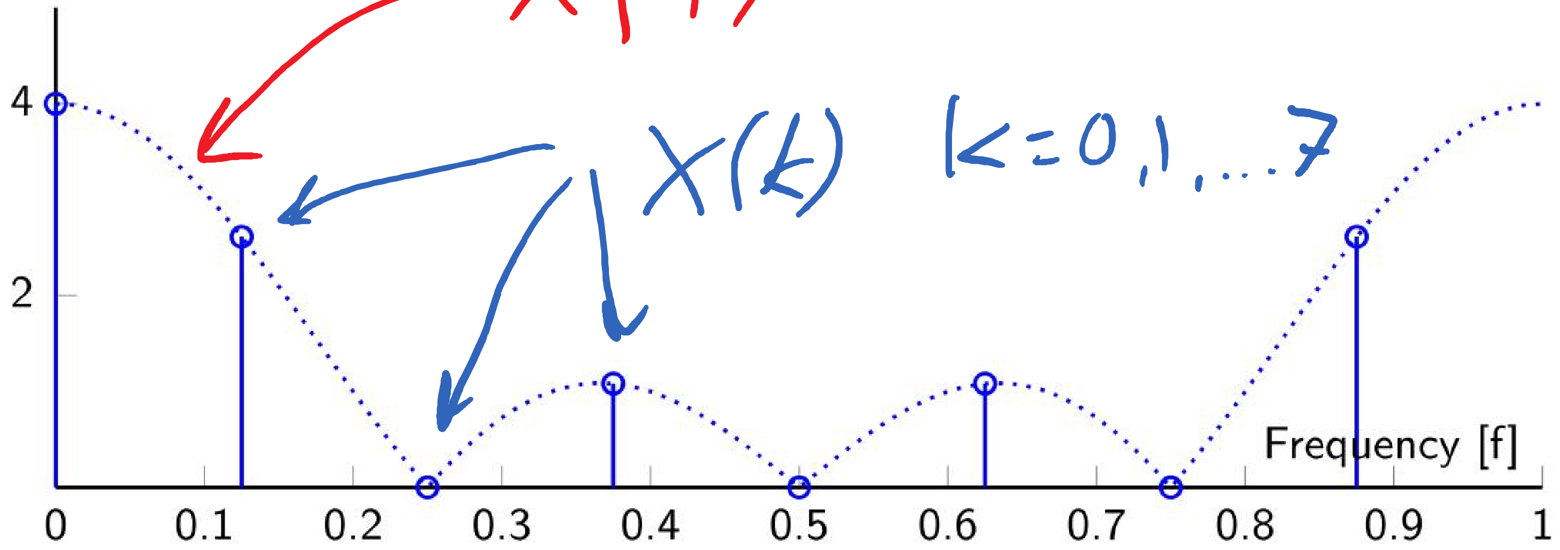
$N = 8$

Let

$$x(n) = \{ \ \dots \ 0 \ \ \underline{1} \ \ 1 \ \ 1 \ \ 1 \ \ 0 \ \ 0 \ \ 0 \ \dots \ \}$$

$X(f)$ $0 < f < 1$

$X(k)$ $k = 0, 1, \dots 7$



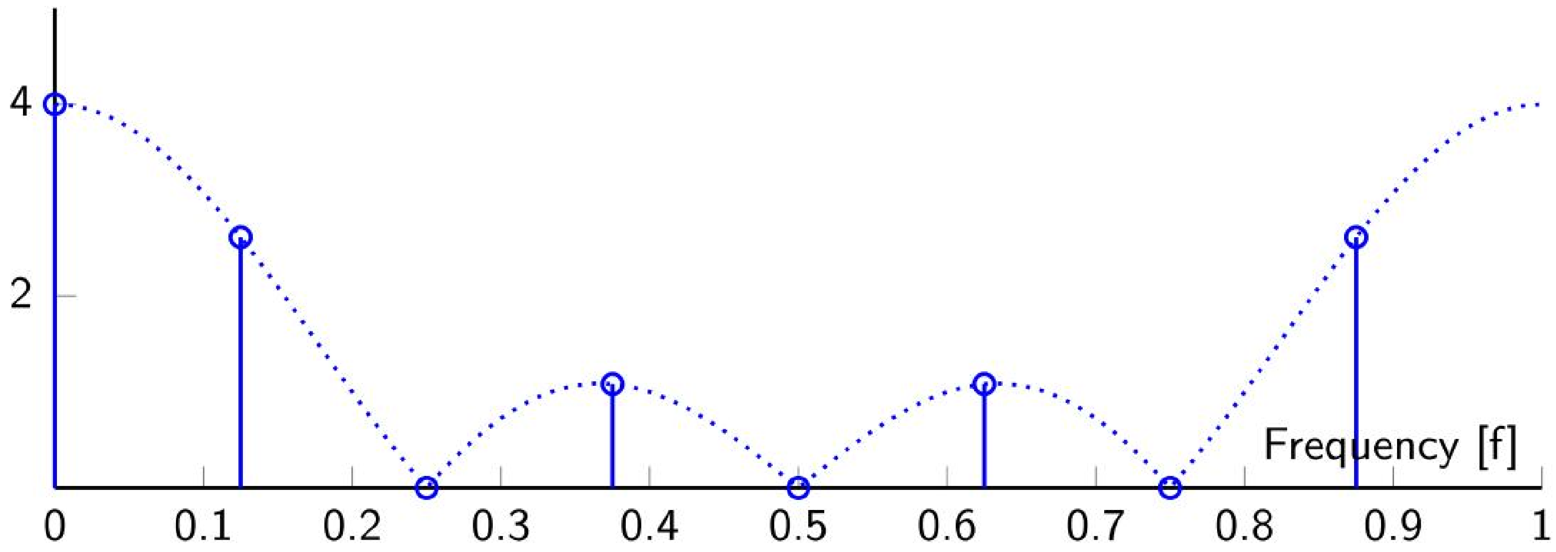
See Lecture 5 slides on page 20 for the example of calculating the Fourier transform of a discrete rectangular pulse [which gives us X(f)]

Conclusion

If $x(n)$ is defined only for $0 \leq n < N$ (length N) we get

$$X_{\text{DFT}}(k) = X(\omega \mid \omega = 2\pi \cdot \frac{k}{N})$$

which is the same as $X(\omega)$ in N uniformly distributed frequencies.



Periodicity

The Fourier transform (DTFT)

$X(\omega)$ is periodic since $e^{j\omega n} = e^{j(\omega+2\pi)n}$.

in ω

The Discrete Fourier transform (DFT)

Both $x(n)$ and $X(k)$ are periodic with period N

since

periodic in n

$$e^{j2\pi\cdot\frac{k}{N}\cdot\overbrace{(n+pN)}} = e^{j2\pi\cdot\frac{k}{N}\cdot n}\cdot \underbrace{e^{j2\pi kp}}_{=1}$$

periodic in k

$$e^{j2\pi\cdot\frac{\overbrace{k+pN}}{N}\cdot n} = e^{j2\pi\cdot\frac{k}{N}\cdot n}\cdot \underbrace{e^{j2\pi kp}}_{=1}$$

$\Rightarrow$ $\boxed{\text{Indices are calculated modulo-}N}$

Ex I Matlab;

N = 4

```
A=-5:1:6;
A=[A;mod(A,4)]

A =

  -5   -4   -3   -2   -1    0    1    2    3    4    5    6
   3    0    1    2    3    0    1    2    3    0    1    2
```

Roots of unity

$$\sum_{k=0}^{N-1} e^{j2\pi\cdot\frac{k}{N}\cdot(n-l)} = \begin{cases} N & \text{om } n-l = 0 + pN \\ 0 & \text{om } n-l \neq 0 + pN \end{cases}$$

$$= N \cdot \delta(n-l, \ \text{mod } N)$$

$$\equiv N \cdot \delta((n-l))_N$$

notation

$$\sqrt[N]{1} = \sqrt[N]{e^{j2\pi k}} =$$

$$|\sqrt[N]{1}| \cdot e^{j2\pi\frac{k}{N}} =$$

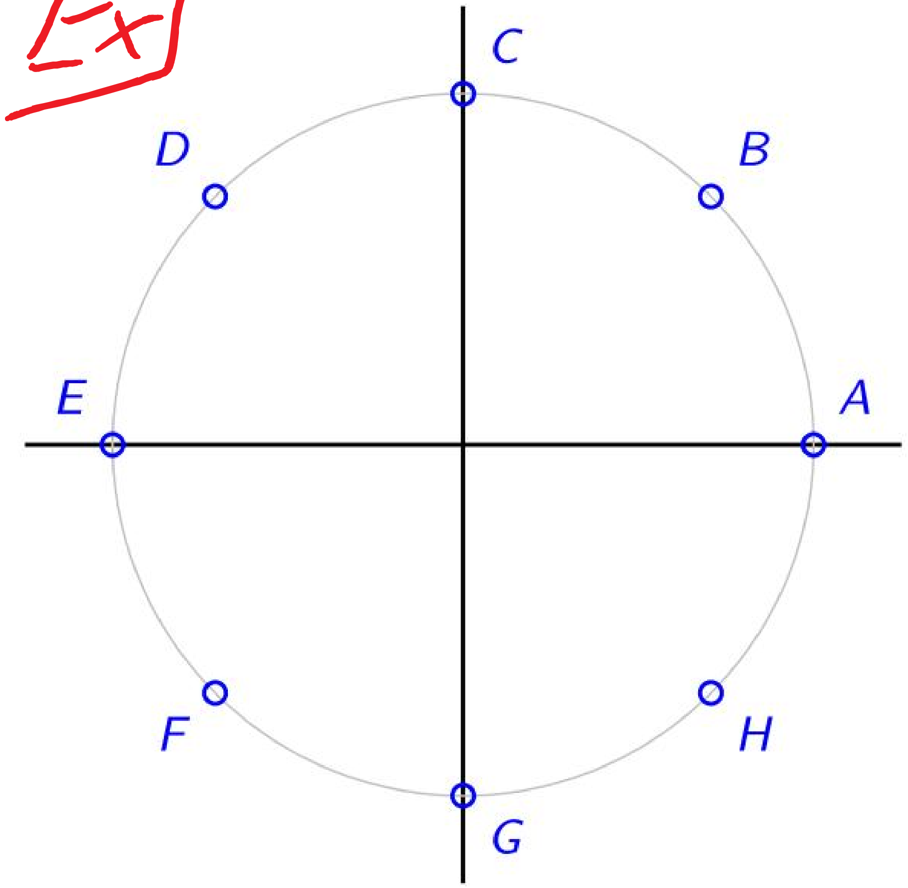
$$e^{j2\pi\frac{k}{N}}, \quad k=0,1,\dots N-1$$

<u>Test:</u> $k=3$

$$\left(e^{j2\pi\frac{3}{N}}\right)^N = e^{j2\pi 3\cdot\frac{N}{N}} = 1$$

$\boxed{\text{Ex}}$

$N = 8$



$$\sum_{k=0}^{N-1} e^{j2\pi\cdot\frac{k}{N}\cdot(n-l)}$$

$$A+A+A+A+A+A+A+A = 8 \quad \text{for } \overset{n-l}{\cancel{k}} = 0 \quad , \pm N \; , \pm 2N \text{ etc}$$

$$A+B+C+D+E+F+G+H = 0 \quad \text{for } \overset{n-l}{\cancel{k}} = 1 \quad \text{— '' —}$$

$$A+C+E+G+A+C+E+G = 0 \quad \text{for } \overset{n-l}{\cancel{k}} = 2\text{, and so on} \quad \text{— '' —}$$

Special properties for the DFT

Both $x(n)$ and $X(k)$ are periodic, which imposes the property that all indices are calculated modulo N.

Ex

$$x(n) = \{\ 3\quad 4\quad \underline{1}\quad 2\quad 3\quad 4\quad 1\quad 2\ \} \tag{19}$$

N = 8

$$x(n-1) = \{\ 2\quad 3\quad \underline{4}\quad 1\quad 2\quad 3\quad 4\quad 1\ \} \tag{20}$$

Circular shift becomes

$$x(n - n_0\,,\ \mathrm{mod}\ N) \quad \Rightarrow \quad X(k)\cdot \mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{k}{N}\cdot n_0}$$

Example of shift for the DFT: N= 4

Ex

$$x(n) = \{\ \underline{1}\quad 2\quad 3\quad 4\ \} \quad \Rightarrow \quad x(n-1) = \{\ \underline{4}\quad 1\quad 2\quad 3\ \}$$

Circular convolution and the DFT, length N (page 476–477)

Multiplication in the DFT-domain is circular convolution in the time domain.

$$X(k) = X_1(k) \cdot X_2(k) \quad \Rightarrow \quad x(n) = x_1(n) \otimes x_2(n) = \sum_{k=0}^{N-1} x_1(k) \cdot x_2(n-k, \text{ mod } N)$$

For this to be valid

Symbol for circular convolution

$x_1(n)$ and $x_2(n)$ needs to have length N

Example

Given:

$$x(n) = \{ \underline{1} \quad 2 \quad 3 \quad 4 \}$$

$$h(n) = \{ \underline{2} \quad 2 \quad 1 \quad 1 \}$$

Find:

$$y_C(n) = x(n) \otimes_{4} h(n)$$

Graphical solution (with $x(n)$ repeated and $h(n)$ time reversed):

$h(k)$	1	1	2	$\underline{2}$	$\rightarrow$					
$x(k)$	2	3	4	$\underline{1}$	2	3	4	1	2	3
$y_C(k)$				$\underline{15}$	13	15	17			

$2 \cdot 2 + 2 \cdot 1 + 1 \cdot 4 + 1 \cdot 3 =$
$4 + 2 + 4 + 3 = 13$

$2 \cdot 1 + 2 \cdot 4 + 3 \cdot 1 + 2 \cdot 1 =$

$2 + 8 + 3 + 2 = 15$

Convolution

Equivalent solution with a table.

$y_c(0) = 15$

$y_c(1) = 13$

$h(n)$ \ $x(n)$	2	3	4	1	2	3	4
				15	13	15	17
2	4	6	8	2	4	6	8
2	4	6	8	2	4	6	8
1	2	3	4	1	2	3	4
1	2	3	4	1	2	3	4

For linear convolution the length of this convolution is 7, but in circular convolution the indices wrap around modulo-4 instead.

```
>> x = [1 2 3 4];
>> h = [2 2 1 1];
>> yc = ifft(fft(x).*fft(h))
yc =
     15     13     15     17
```

equivelent if

length(x) = length(h) = N

We can also use the function cconv(.)

```
>> cconv(x,h,4)

ans =

   15   13   15   17
```

$\boxed{Ex}$ $x_1(n) = \begin{bmatrix} 1 & \underset{\uparrow}{1} & 2 & 2 & 3 \end{bmatrix}$

$x_2(n) = \begin{bmatrix} \underset{\uparrow}{1} & -1 \end{bmatrix}$

find $y(n) = x_1(n) \otimes_4 x_2(n)$

Solution

we had $x_1(n) = [1\ 1\ 2\ 2\ 3]$
$x_2(n) = [1\ -1]$

First make $x_1(n)$ and $x_2(n)$ to N-length sequences, by truncation or by adding zeroes.

in mod 4

$$\Rightarrow x_1(n) = [1\ 1\ 2\ 2] = [1\ 2\ 2\ 1]$$

$$\Rightarrow x_2(n) = [1\ -1\ 0\ 0]$$

$$\Rightarrow \quad y(n) = x_1(n) \otimes_4 x_2(n)$$

Or with a table

$x_2 \backslash x_1$	2	2	1	1	2	2	1
1	2	2	1	1	2	2	1
-1	-2	-2	-1	-1	-2	-2	-1
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

<u>In Matlab we can now write;</u>

```
y=ifft(fft([1 2 2 1]).*fft([1 -1 0 0]))
y =

    0    1    0   -1
```

OR

```
y=cconv([1 2 2 1],[1 -1 0 0],4)
y =

    0    1    0   -1
```

$$\Rightarrow \quad y(n) = x_1(n) \circledast_4 x_2(n)$$

Or with a table

$x_2 \backslash x_1$	2	2	1	1	2	2	1
1	2	2	1	1	2	2	1
-1	-2	-2	-1	-1	-2	-2	-1
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

=0 =1 =0 =-1

$$\Rightarrow y(n) = [0 \;\; 1 \;\; 0 \;\; -1]$$

<u>In Matlab we can now write;</u>

```
y=ifft(fft([1 2 2 1]).*fft([1 -1 0 0]))
y =

    0    1    0   -1
```

OR

```
y=cconv([1 2 2 1],[1 -1 0 0],4)
y =

    0    1    0   -1
```

Linear convolution and the DFT

The convolution between $x(n)$ and $h(n)$ yields $y(n)$ of length $4+4-1$. Choose a DFT length of $N=8$.

$h(k)$	1	1	2	$\underline{2}$	$\rightarrow$									
$x(k)$	0	0	0	$\underline{1}$	2	3	4	0	0	0	0	1	2	3
$y_L(k)$				$\underline{2}$	6	11	17	13	7	4	0			

```
>> x = [1 2 3 4];
>> h = [2 2 1 1];
>> yl = ifft(fft(x,8).*fft(h,8))
yl =
     2     6     11     17     13     7     4     0
```

% or we can use: cconv(x,h,8)

```
>> conv([1 2 3 4],[2 2 1 1])
ans =   2    6    11    17    13    7    4
```

linear convolution

The linear convolution is closely related to the circular convolution.

Ex cont

N = 4

$$y_C(n) = \{\ y_L(0)+y_L(4)\quad y_L(1)+y_L(5)\quad y_L(2)+y_L(6)\quad y_L(3)+y_L(7)\ \}$$

$$= \{\ 2+13\quad 6+7\quad 11+4\quad 17+0\ \}$$

$$= \{\ 15\quad 13\quad 15\quad 17\ \}$$

we had

yL =	2	6	11	17	13	7	4	0
	n = 0	1	2	3	4	5	6	7

Sampling the spectrum

Let

DTFT

$$x(n) = a^n \cdot u(n) \quad \Rightarrow \quad X(\omega) = \frac{1}{1 - a e^{-j\omega}}$$

sample i.e.

$\omega \longmapsto 2\pi \frac{k}{N}$

IDFT

$$x(n) = ? \quad \Longleftarrow \quad X(k) = \frac{1}{1 - a e^{-j2\pi \cdot \frac{k}{N}}}$$

Inverse-DFT yields:

$$x_{\mathrm{DFT}}(n) = \overbrace{\frac{1}{N}\cdot\sum_{k=0}^{N-1} X(k)\mathrm{e}^{\mathrm{j}2\pi\cdot\frac{n}{N}\cdot k}}^{\text{IDFT}}$$

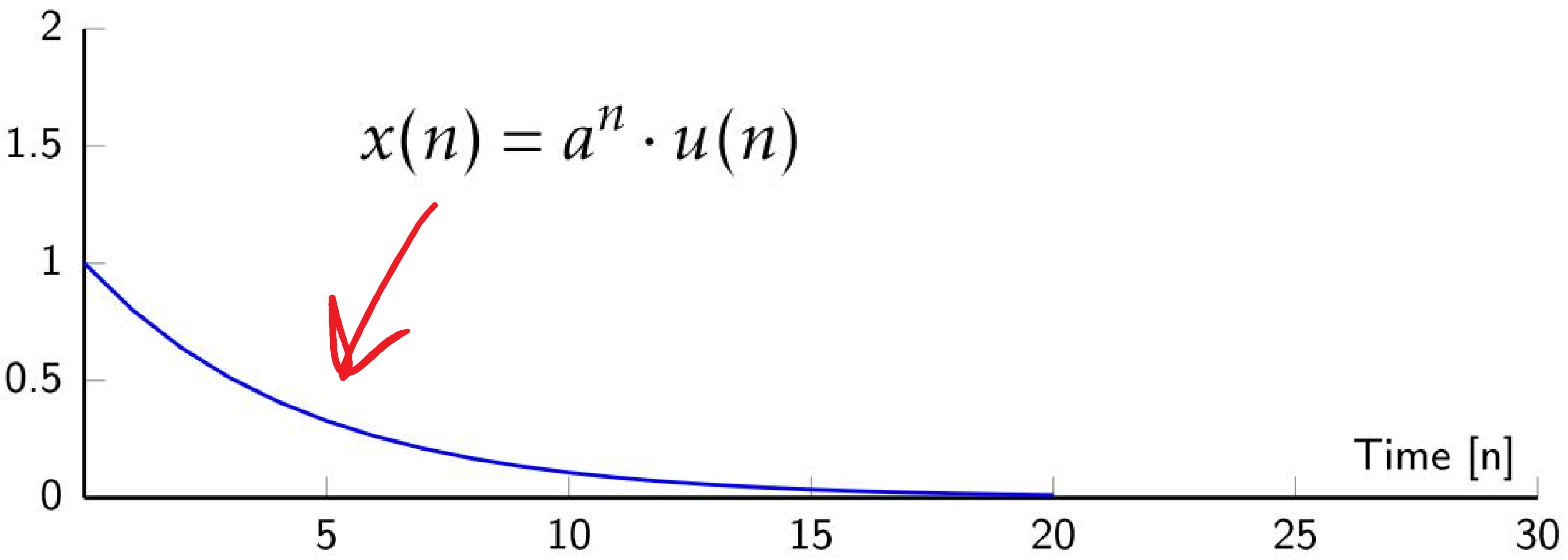
$$= \frac{1}{N}\cdot\sum_{k=0}^{N-1} \overbrace{\sum_{m=-\infty}^{\infty} x(m)\mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{m}{N}\cdot k}}^{\mathrm{DTFT}\{x(m)\}\big|_{\omega \mapsto 2\pi\frac{k}{N}}}\cdot\mathrm{e}^{\mathrm{j}2\pi\cdot\frac{n}{N}\cdot k}$$

$$= \frac{1}{N}\cdot\sum_{m=-\infty}^{\infty} x(m)\cdot\underbrace{\boxed{\sum_{k=0}^{N-1}\mathrm{e}^{\mathrm{j}2\pi\cdot\frac{n-m}{N}\cdot k}}}_{\text{roots of unity}}$$

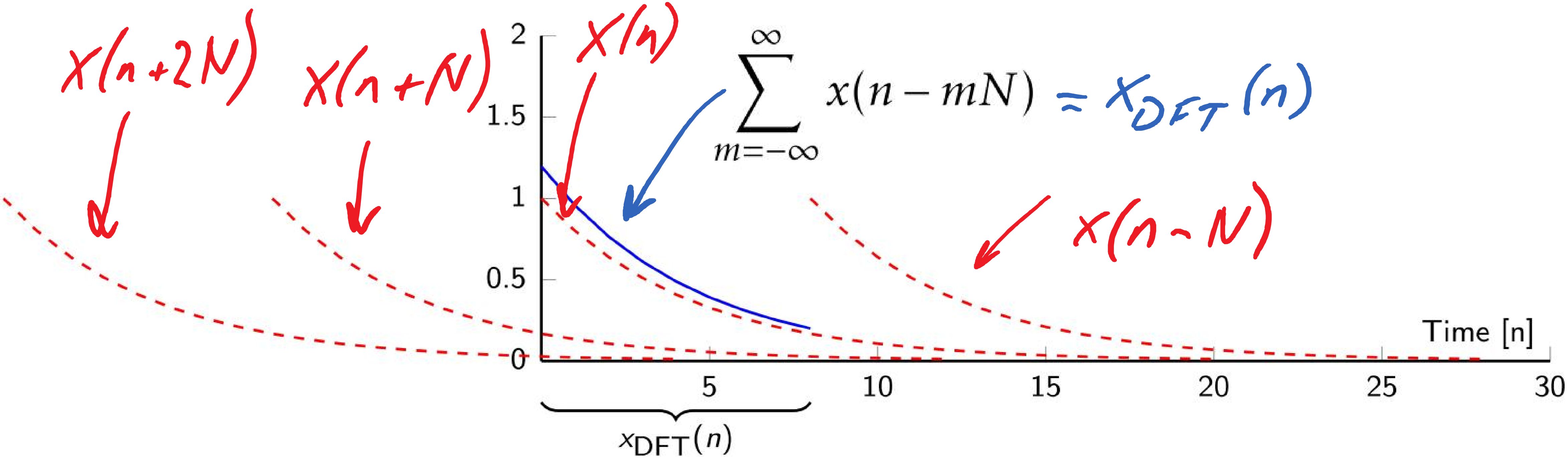
$$= \frac{1}{N}\cdot\sum_{m=-\infty}^{\infty} x(m)\cdot N\cdot\delta(n-m\,,\ \mathrm{mod}\ N)$$

$$= \sum_{m=-\infty}^{\infty} x(n-mN)$$

The signal $x(n)$ extends to $+\infty$.



The signal $x_{\mathrm{DFT}}(n)$ is the sum of all shifted and repeated $x(n)$. The DFT size is $N = 8$.



Try some Matlab examples of the above effect as given in the lecture notes on page 8

A practical application

Increase the resolution in frequency by *zero padding* or *trailing zeros*.

Let

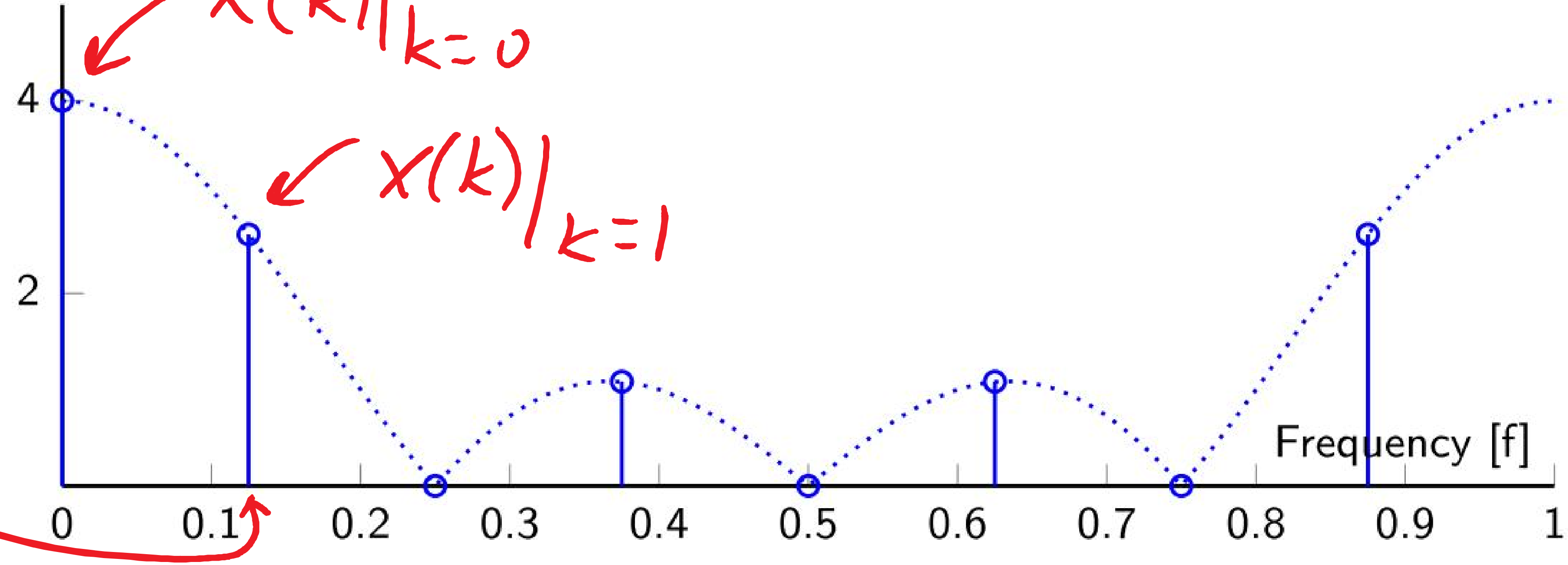
$$x(n) = \{\ \underline{1}\quad 1\quad 1\quad 1\quad 0\quad 0\quad \ldots\ \}$$

Calculate the DFT of length $N = 8$ of $x(n)$.

In matlab; X=fft([1 1 1 1],8)

$X(k)\big|_{k=0}$

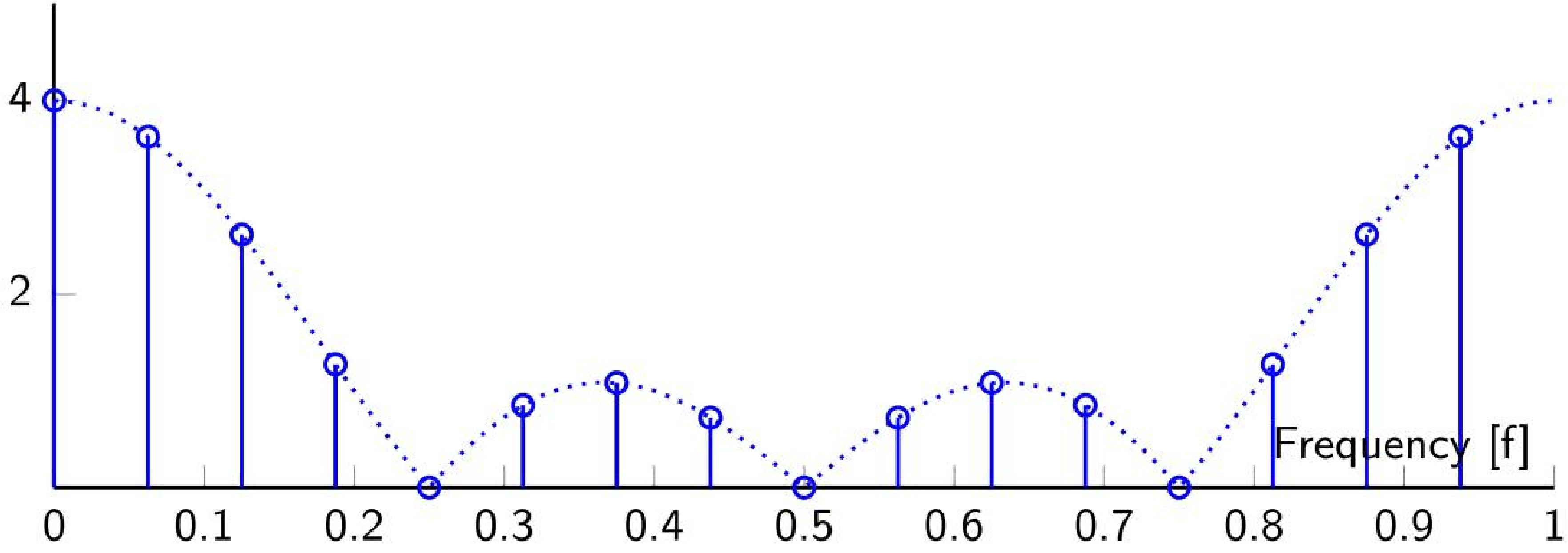
$X(k)\big|_{k=1}$



$f = \frac{1}{N} = \frac{1}{8}$

Calculate the DFT of length $N = 16$ of $x(n)$.

In matlab; X=fft([1 1 1 1],16)



The DFT on matrix form (page 459–460)

Define

$$W_N = e^{-j2\pi/N} = W$$

The DFT and the IDFT becomes

$$X(k) = \sum_{n=0}^{N-1} x(n) W^{kn}$$

DFT

$$x(n) = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k) W^{-kn}$$

IDFT

Let

$$\mathbf{x} = \begin{bmatrix} x(0) & x(1) & \cdots & x(N-1) \end{bmatrix}^{\mathrm{T}}$$

vector $(N \times 1)$

and

$$\mathbf{X} = \begin{bmatrix} X(0) & X(1) & \cdots & X(N-1) \end{bmatrix}^{\mathrm{T}}$$

vector $(N \times 1)$

and

$$\mathbf{D} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W & W^2 & \cdots & W^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W^{N-1} & W^{2(N-1)} & \cdots & W^{(N-1)(N-1)} \end{bmatrix}$$

matrix $(N \times N)$

Ex In Matlab;

W=dftmtx(4) $N=4$

W =

```
   1.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   1.0000 + 0.0000i   0.0000 - 1.0000i  -1.0000 + 0.0000i   0.0000 + 1.0000i
   1.0000 + 0.0000i  -1.0000 + 0.0000i   1.0000 + 0.0000i  -1.0000 + 0.0000i
   1.0000 + 0.0000i   0.0000 + 1.0000i  -1.0000 + 0.0000i   0.0000 - 1.0000i
```

We can ~~not~~ now describe the DFT and the IDFT as

$$\mathbf{X} = \mathbf{D}\mathbf{x}$$ DFT

and

$$\mathbf{x} = \mathbf{D}^{-1}\mathbf{X}$$ IDFT

Discrete

Prove the Fourier-transf!

$$x = D^{-1} X = \underbrace{D^{-1} \cdot D}_{= I} \cdot x = x$$