Lecture 11

Digital Signal Processing

Chapter 7

Discrete Fourier Transform
DFT

From lecture 10 we had,

The definition of the DFT

$$X_{\mathrm{DFT}}(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi\cdot\frac{k}{N}\cdot n}$$

$k = 0, 1, 2, \ldots N-1$

$$X_{\mathrm{IDFT}}(\overset{n}{\cancel{k}}) = \frac{1}{N}\cdot\sum_{\underset{k}{\cancel{n}}=0}^{N-1} \overset{X(k)}{\cancel{x(n)}}e^{j2\pi\cdot\frac{k}{N}\cdot n}$$

$n = 0, 1, 2, \ldots N-1$

Both $x(n)$ and $X(k)$ are periodic and indices are calculated modulo-N.

Circular shift:

$$y(n) = x(n - n_0, \bmod N) \quad \Rightarrow \quad Y(k) = e^{-j2\pi\cdot\frac{k}{N}n_0}\cdot X(k)$$

Circular convolution:

$$y(n) = x_1(n) \otimes_N x_2(n) = \sum_{k=0}^{N-1} x_1(n)x_2(n-k, \bmod N) \quad \Rightarrow \quad Y(k) = X_1(k)\cdot X_2(k)$$

Example of DFT

length

The Fourier transform of an infinite signal:

$$x(n) = a^n u(n) \quad \Rightarrow \quad X(\omega) = \frac{1}{1 - a\mathrm{e}^{-\mathrm{j}\omega}}$$

The Fourier transform of a finite signal:

length

$$x(n) = a^n \quad \text{for } 0 \leq n < N \quad \Rightarrow \quad X(\omega) = \frac{1 - a^N \mathrm{e}^{-\mathrm{j}\omega N}}{1 - a\mathrm{e}^{-\mathrm{j}\omega}}$$

$\omega = 2\pi \frac{k}{N}$

The discrete Fourier transform of a finite signal:

length

$$x(n) = a^n \quad \text{for } 0 \leq n < N \quad \Rightarrow \quad X(k) = \frac{1 - a^N}{1 - a\mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{k}{N}}}$$

How can we use DFT and IDFT to perform the linear filtering operation for an infinite length input signal?

Answer: By using the Overlap-Add method

Filtering with the Overlap-Add, (page 487,488)

- In a real time environment the signal is streaming and is never available as a whole. The signal has no start and no end.

In overlap-add the signal is divided into blocks of length L samples and each block is filtered independently.

$$x(n) = \underbrace{\boxed{x_1(n)\qquad\qquad}}_{L}\underbrace{\boxed{x_2(n)\qquad\qquad}}_{L}\underbrace{\boxed{x_3(n)\qquad\qquad}}_{L}\cdots$$

The length of the impulse response is M so zero-pad each block to length $N = L+M-1$.

$$\begin{array}{lllll}
\boxed{x_1(n)\qquad\qquad} & \boxed{0\quad} & & & \\
 & \boxed{x_2(n)\qquad\qquad} & \boxed{0\quad} & & \\
 & & \boxed{x_3(n)\qquad\qquad} & \boxed{0\quad} &
\end{array}$$

Filter each zero-padded block individually.

$$\begin{array}{lllllll}
\boxed{\uparrow\quad} & \boxed{y_1(n)\qquad} & \boxed{\downarrow\quad} & & & & \\
 & & \boxed{\uparrow\quad} & \boxed{y_{\cancel{1}\,2}(n)\qquad} & \boxed{\downarrow\quad} & & \\
 & & & & \boxed{\uparrow\quad} & \boxed{y_{\cancel{1}\,3}(n)\qquad} & \boxed{\downarrow\quad}
\end{array}$$

Example

L=4, M=4

Given: The input signal

$$x(n) = \{ \underbrace{1 \quad 0 \quad 1 \quad 1}_{x_1} \quad \underbrace{1 \quad 0 \quad 1 \quad 0}_{x_2} \quad \underbrace{0 \quad 1 \quad 0 \quad 1}_{x_3} \}$$

and the filter

$$h(n) = \{ 1 \quad 1 \quad 0 \quad 0 \}$$

Length = M = 4

```
>> conv([1 0 1 1 1 0 1 0 0 1 0 1],[1 1 0 0])

ans =   1   1   1   2   2   1   1   1   0   1   1   1   1   0   0
```

Find: The block-filtering $y(n) = x(n) * h(n)$ with $L = 4$, $M = 4$ and $N = L + M - 1 = 7$

Solution:

$x_1 * h =$	1	1	1	2	1	0	0								
$x_2 * h =$					1	1	1	1	0	0	0				
$x_3 * h =$									0	1	1	1	1	0	0
$x * h =$	1	1	1	2	2	1	1	1	0	1	1	1	1	0	0

```
>> ifft(fft([1 0 1 1],7).*fft([1 1 0 0],7))

ans =

   1.0000    1.0000    1.0000    2.0000    1.0000   -0.0000    0.0000
```

DFT of a sine (whole number of periods)

Give~~t~~n:

$$x(n) = \cos\left(2\pi \cdot \frac{k_0}{N} \cdot n\right)$$

→ periodic by N
(period ≤ N)

Find: The discrete Fourier transform $X(k)$ of $x(n)$.

Solution:

$$X(k)=\sum_{n=0}^{N-1} x(n)\mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{k}{N}\cdot n}$$

Eulers

$$=\sum_{n=0}^{N-1}\frac{1}{2}\cdot\left[\mathrm{e}^{\mathrm{j}2\pi\cdot\frac{k_0}{N}\cdot n}+\mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{k_0}{N}\cdot n}\right]\cdot\mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{k}{N}\cdot n}$$

$$=\sum_{n=0}^{N-1}\frac{1}{2}\cdot\mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{k-k_0}{N}\cdot n}+\sum_{n=0}^{N-1}\frac{1}{2}\cdot\mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{k+k_0}{N}\cdot n}$$

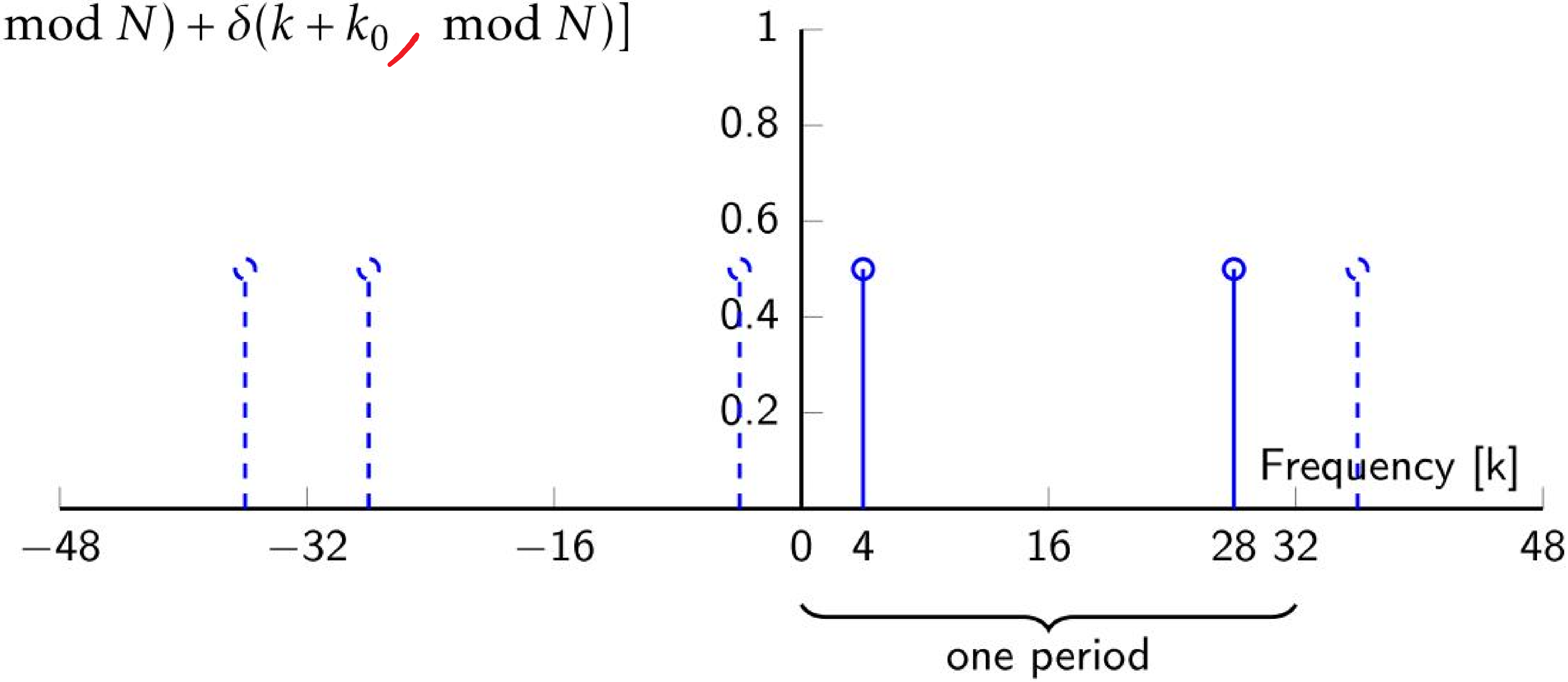
roots of unity

$$=\frac{N}{2}\cdot[\delta(k-k_0,\ \mathrm{mod}\ N)+\delta(k+k_0,\ \mathrm{mod}\ N)]$$

The spe~~k~~ctrum for $k_0 = 4$ and $N = 32$.

$$\frac{N}{2} \cdot [\delta(k - k_0, \bmod N) + \delta(k + k_0, \bmod N)]$$

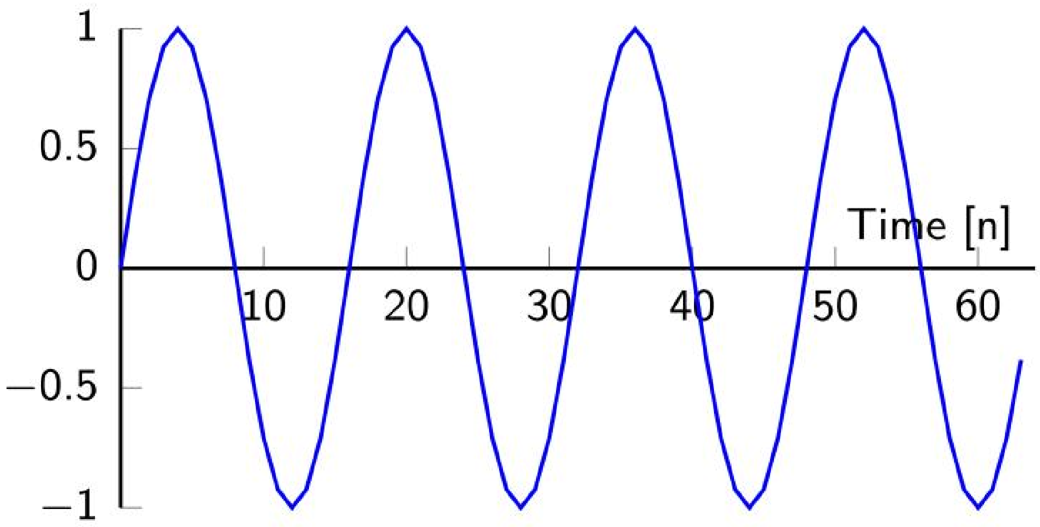
x32



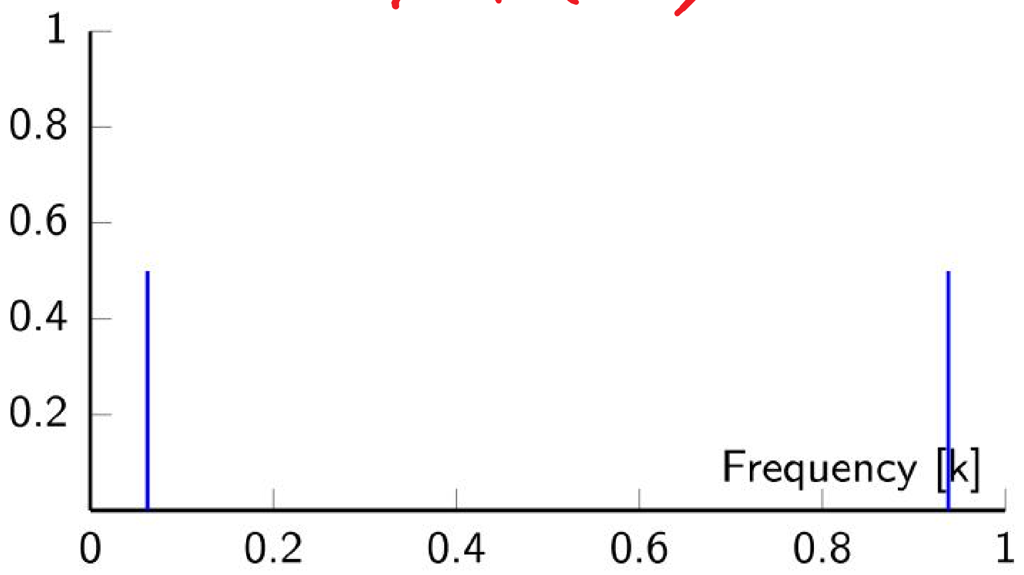
Examples of the effect that $x(n)$ is periodic for the DFT

The DFT of a sine with a whole number of periods:

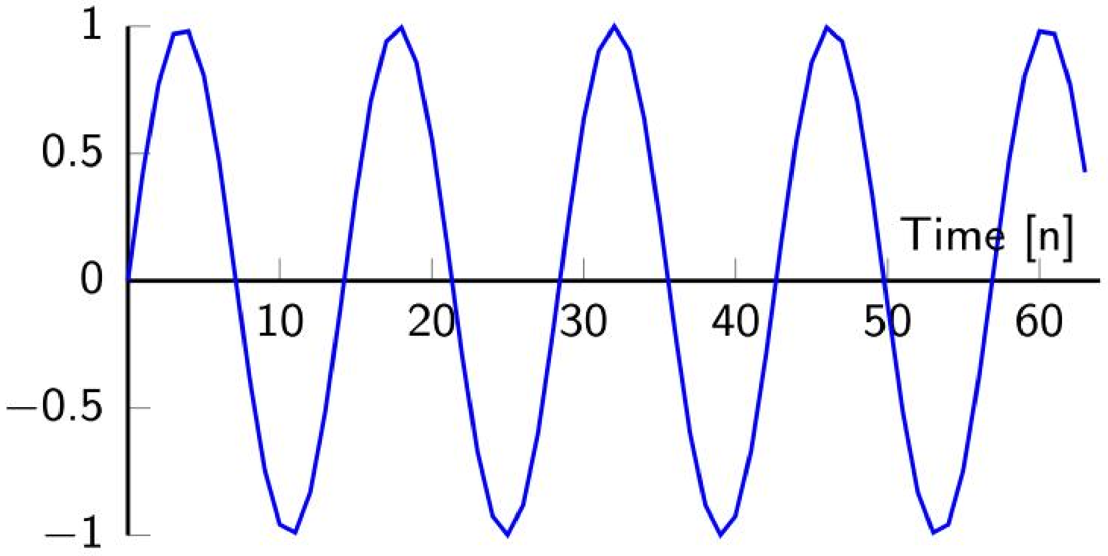
$X(n)$



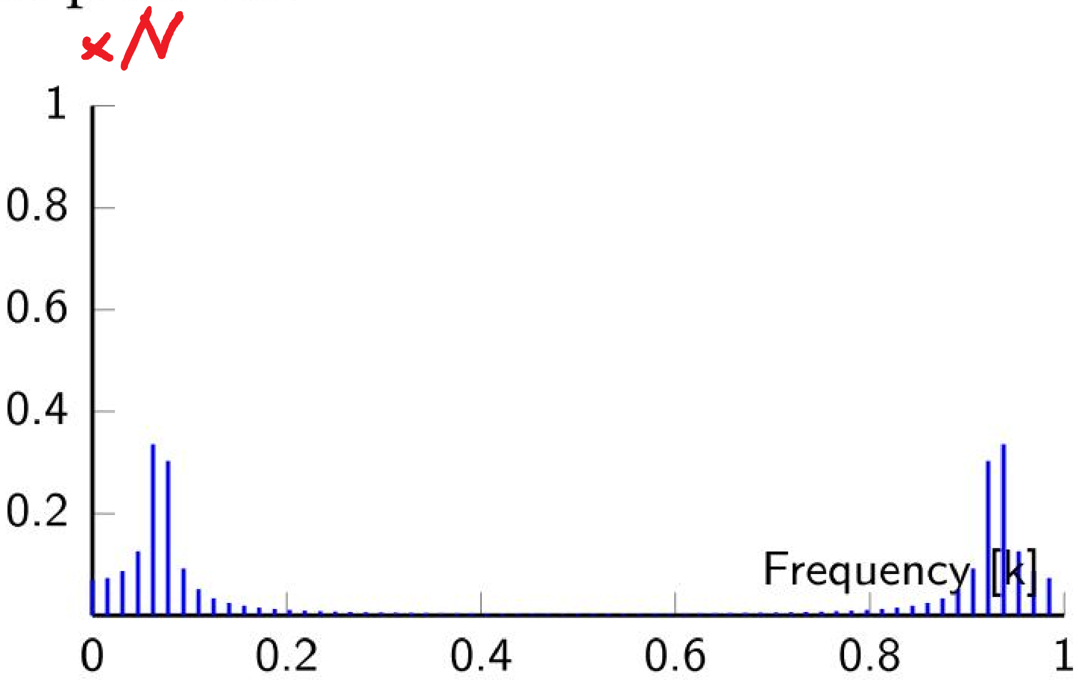
$\times N$ $X(k)$



The DFT of a sine with a fractional number of periods:

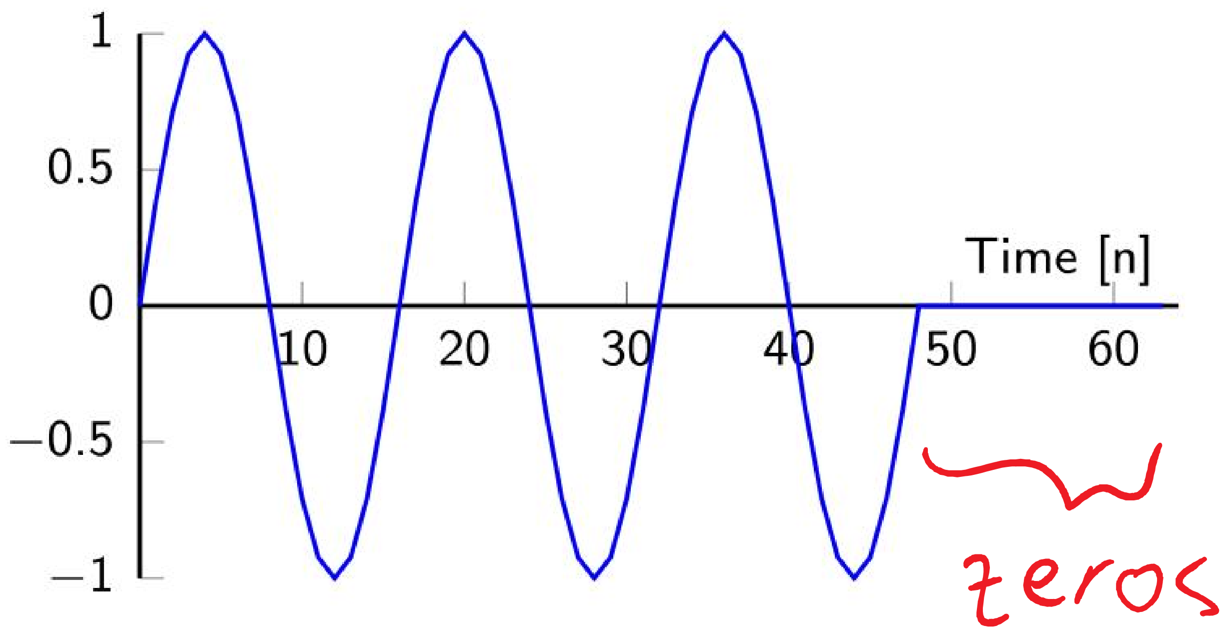


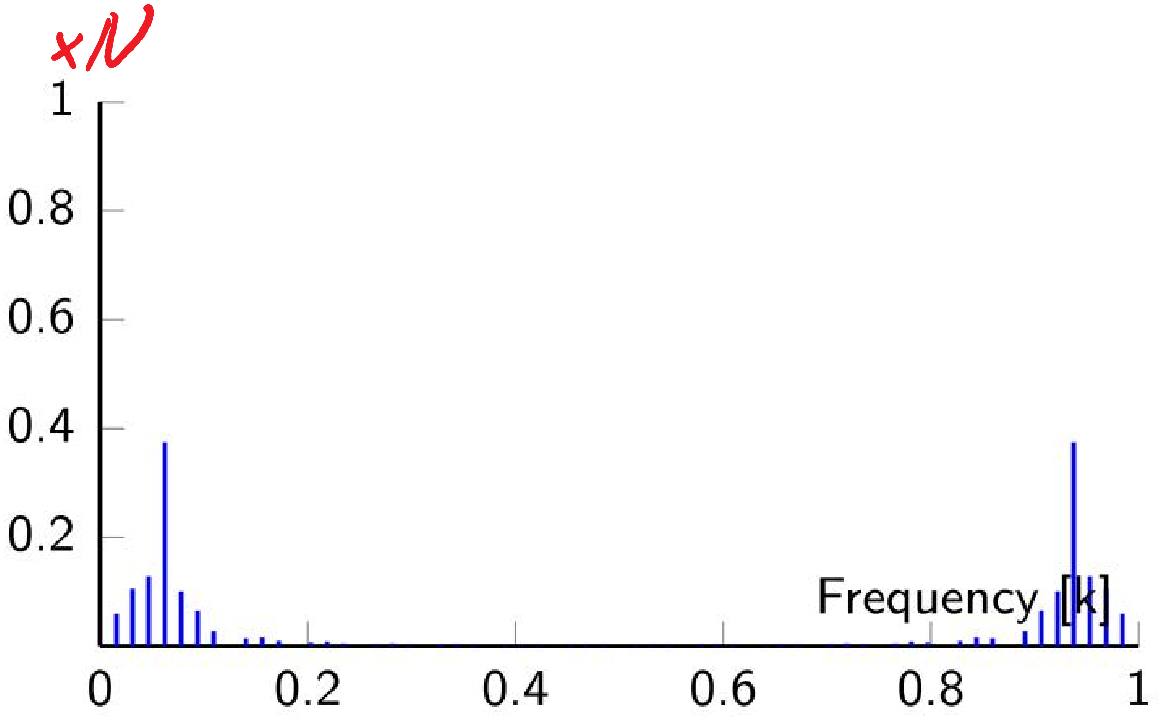
$x(n)$



$X(k)$

The DFT of a sine with zero padding:





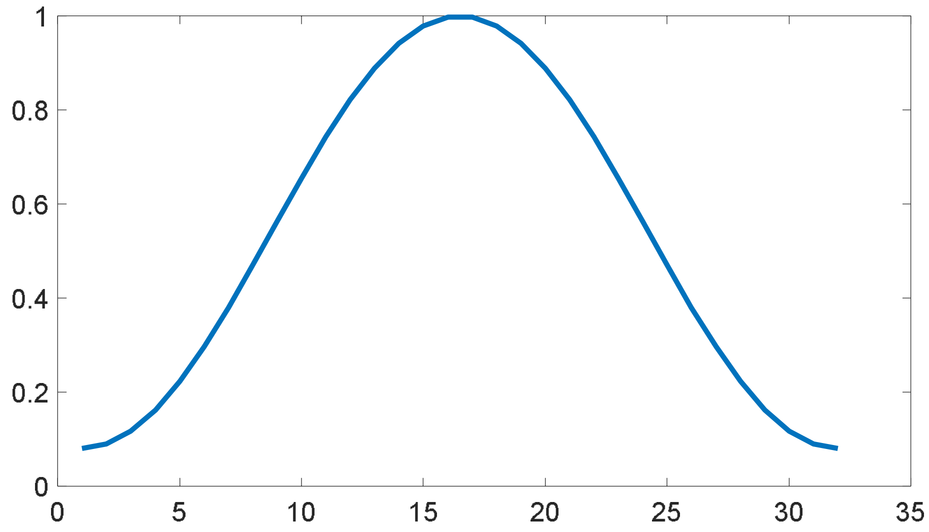
Reduce the effect of truncation with a time window

By multiplying the signal with a window that attenuates the signal near the ends the effect of the truncation can be reduced:

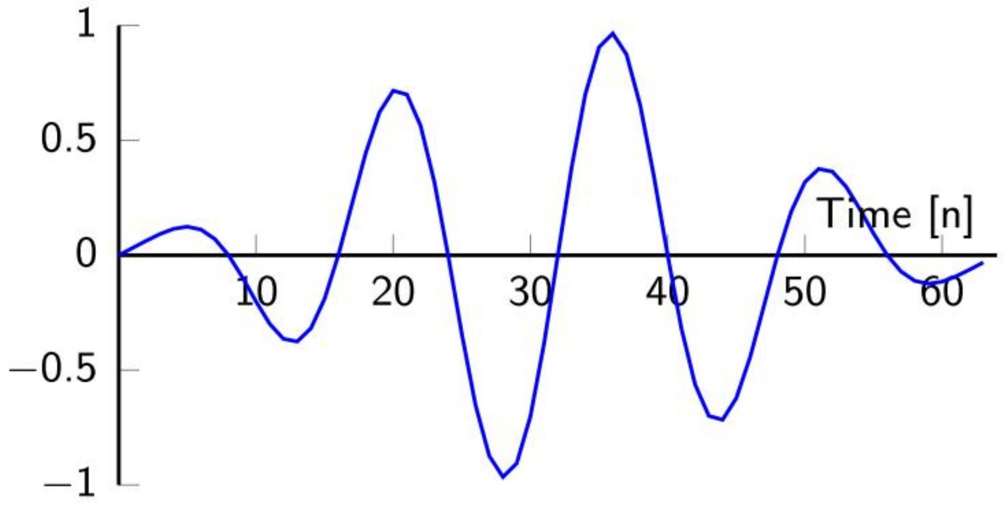
$$x_w(n) = x(n) \cdot w(n) \tag{17}$$

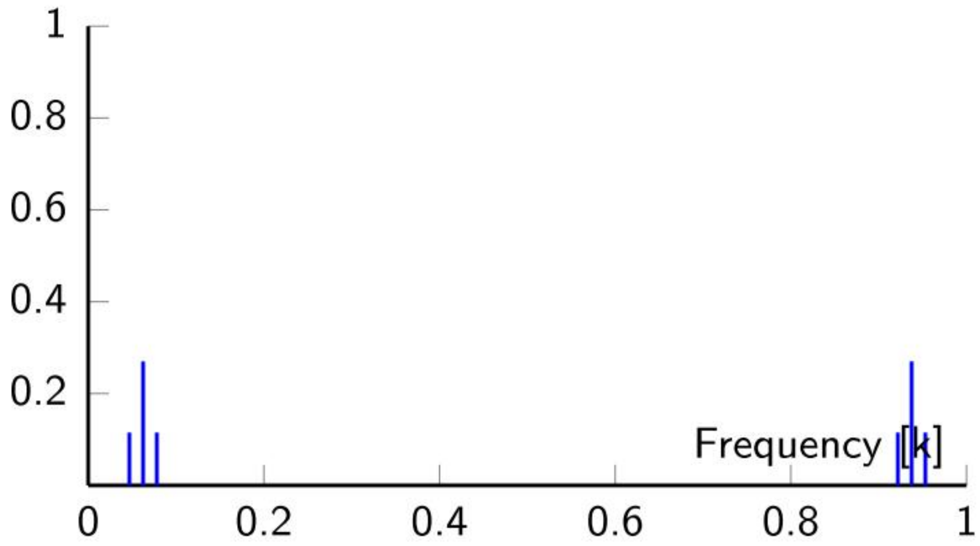
for example, a Hamming windows defined as

$$w_{\text{hamming}}(n) = 0.54 + 0.46 \cos\left(2\pi \cdot \frac{1}{N-1} \cdot \left(n - \frac{N-1}{2}\right)\right)$$

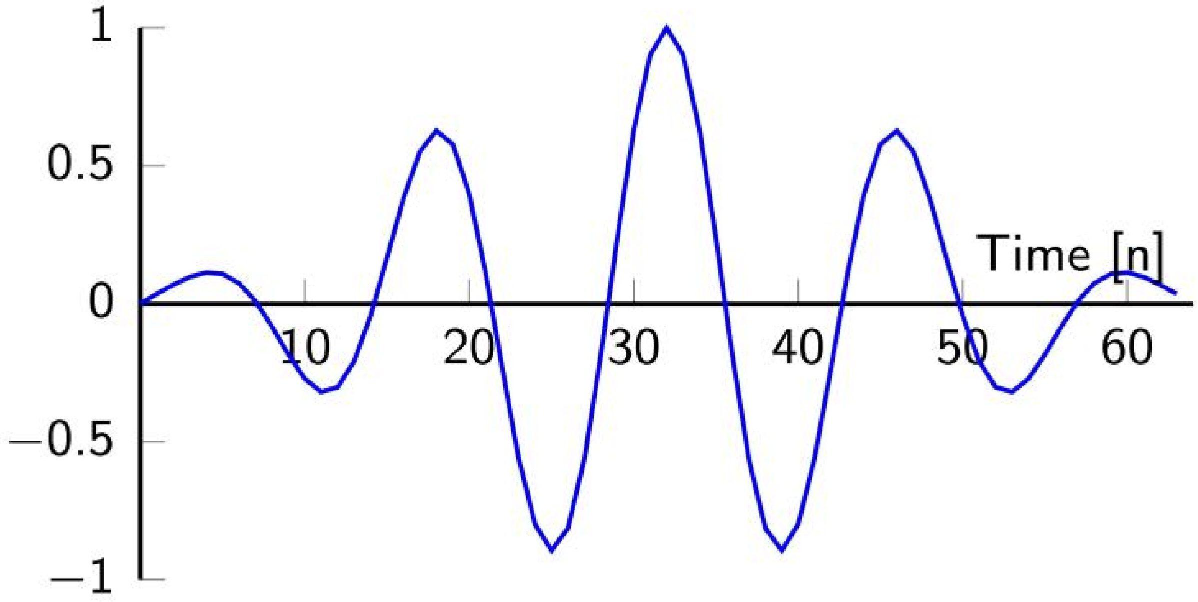


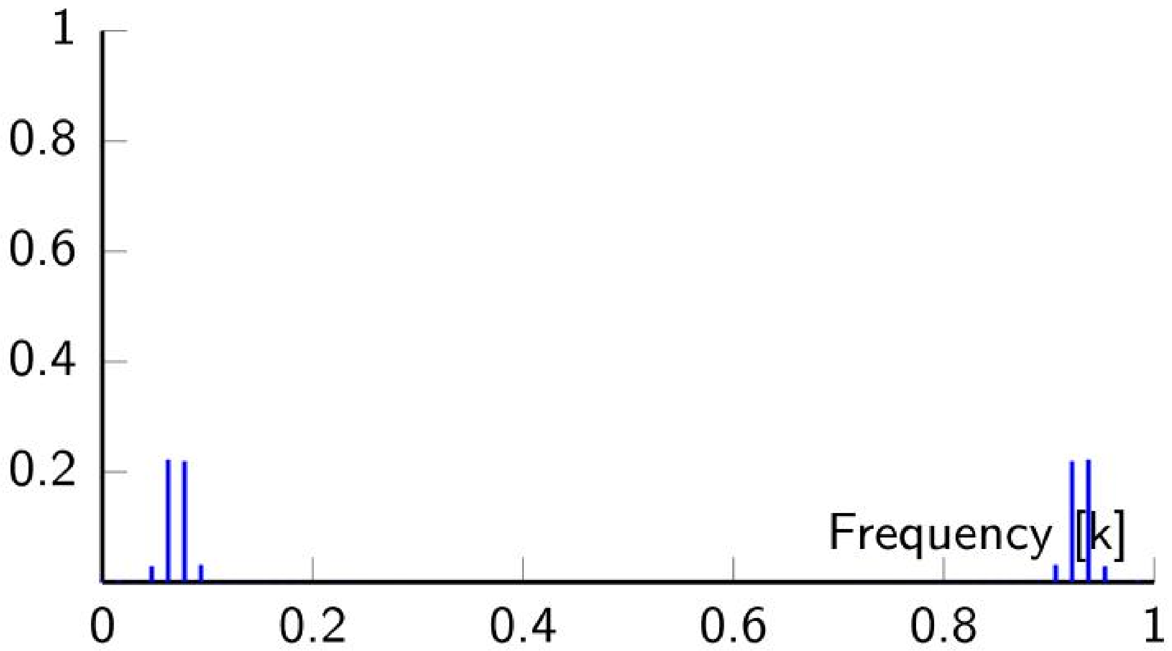
The DFT of a windowed sine with a whole number of periods:





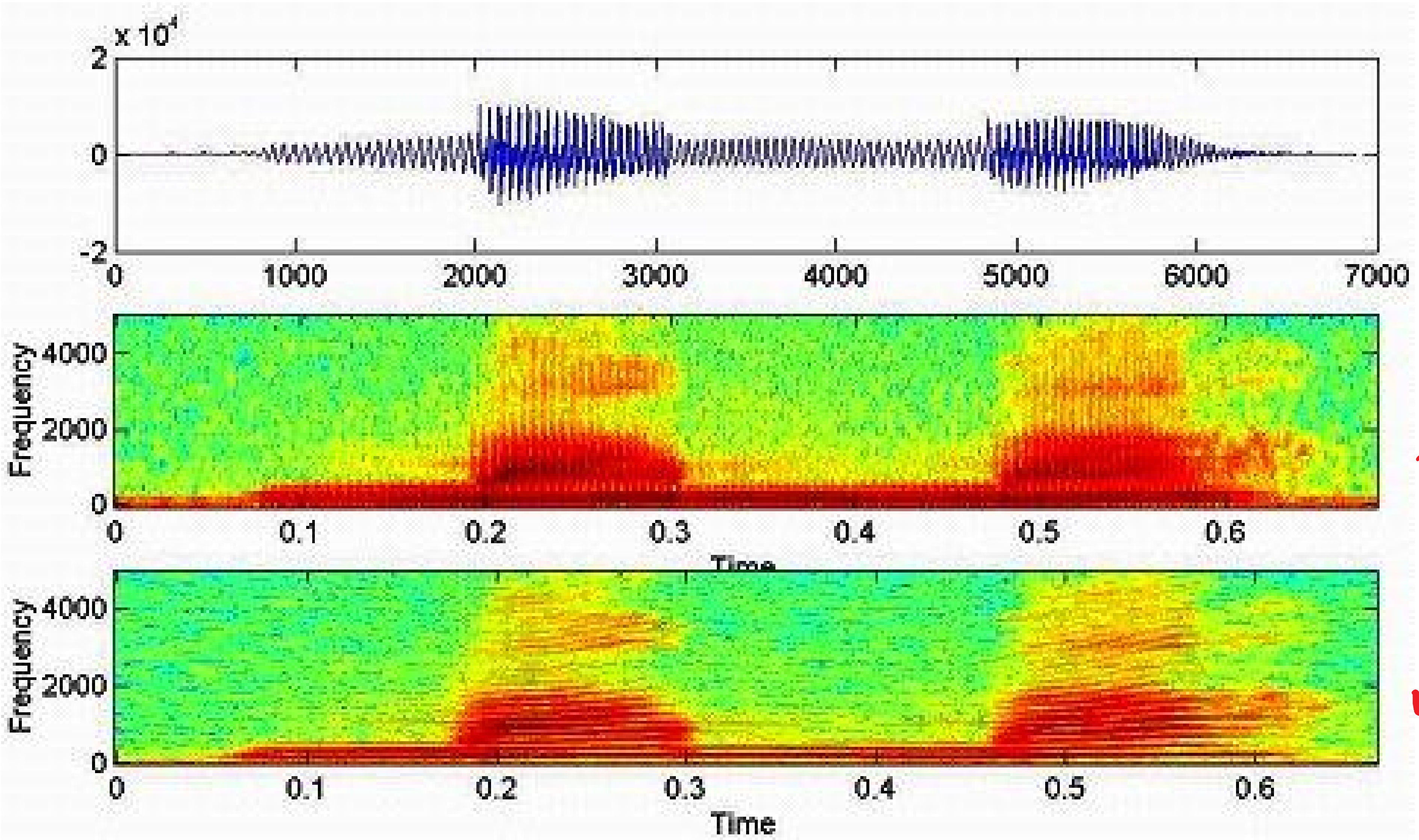
The DFT of a windowed sine with a fractional number of periods:





Spectrogram: spectrum as a function of time

An acoustic recording of the word “mama”



short window

long window

Ex: Scalogram using Wavelets (from Matlab)