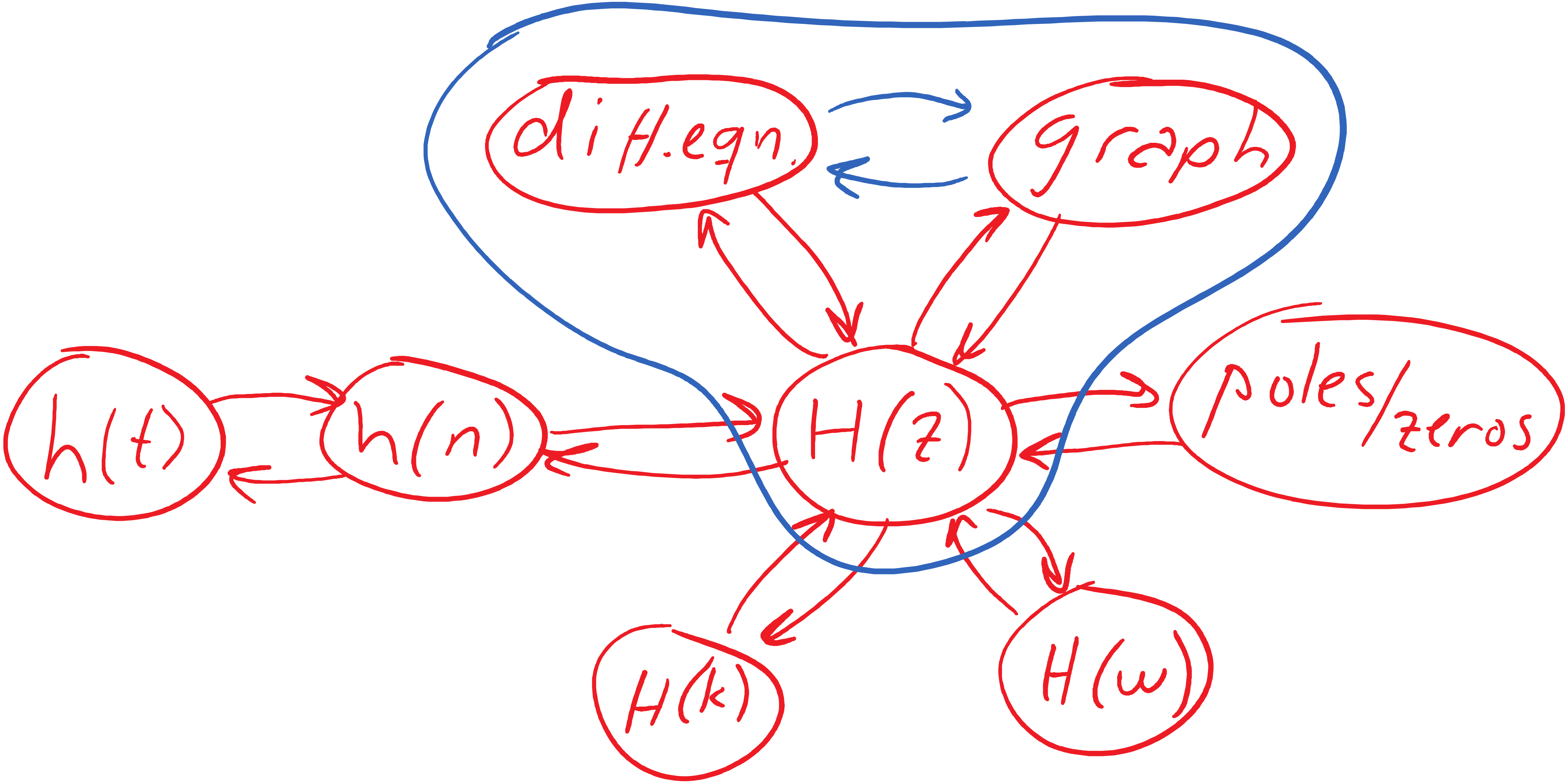
Lecture 12

Digital Signal Processing

Chapter 9

Structures (= graphs)


diff.eqn.
graph
h(t)
h(n)
H(z)
poles/zeros
H(k)
H(ω)

Difference equations

FIR

$$y(n) = \sum_{k=0}^{M} b_k x(n-k)$$

+ Always stable.
+ Can be made with linear phase if $h(n)$ is symmetric.
– The order M is often large (more computationally demanding).
– Non-parametric (for example, difficult to describe resonance).

Difference equations

IIR

$$y(n)+\sum_{k=1}^{N}a_k y(n-k)=\sum_{k=0}^{M}b_k x(n-k) \tag{2}$$

+ The numerator and denominators orders M and N can be made small (less computationally demanding).

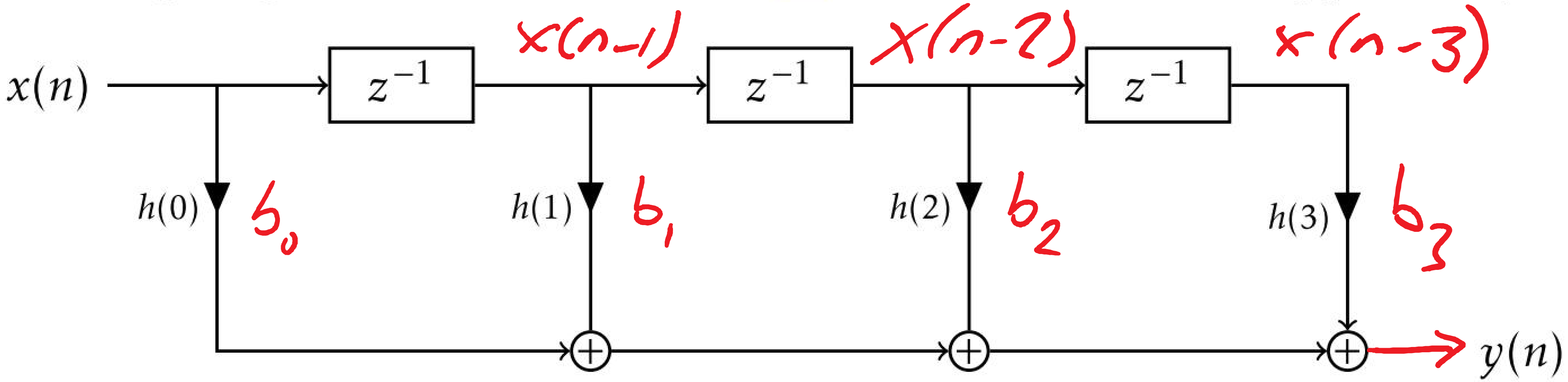
+ Parametric (for example, poles describe resonance).

– Can be unstable.

– Cannot have linear phase.

FIR filters

The following diagram is called direct form, transversal filter, or tapped delay filter.



From the figure we can immediately identify

$$y(n) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + h(3)x(n-3) = \sum_{k=0}^{3} h(k)x(n-k)$$

$$= \sum_{k=0}^{3} b_k x(n-k)$$

We had from the picture;

$$y(n) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + h(3)x(n-3) = \sum_{k=0}^{3} h(k)x(n-k)$$

and taking the Z-transform we get

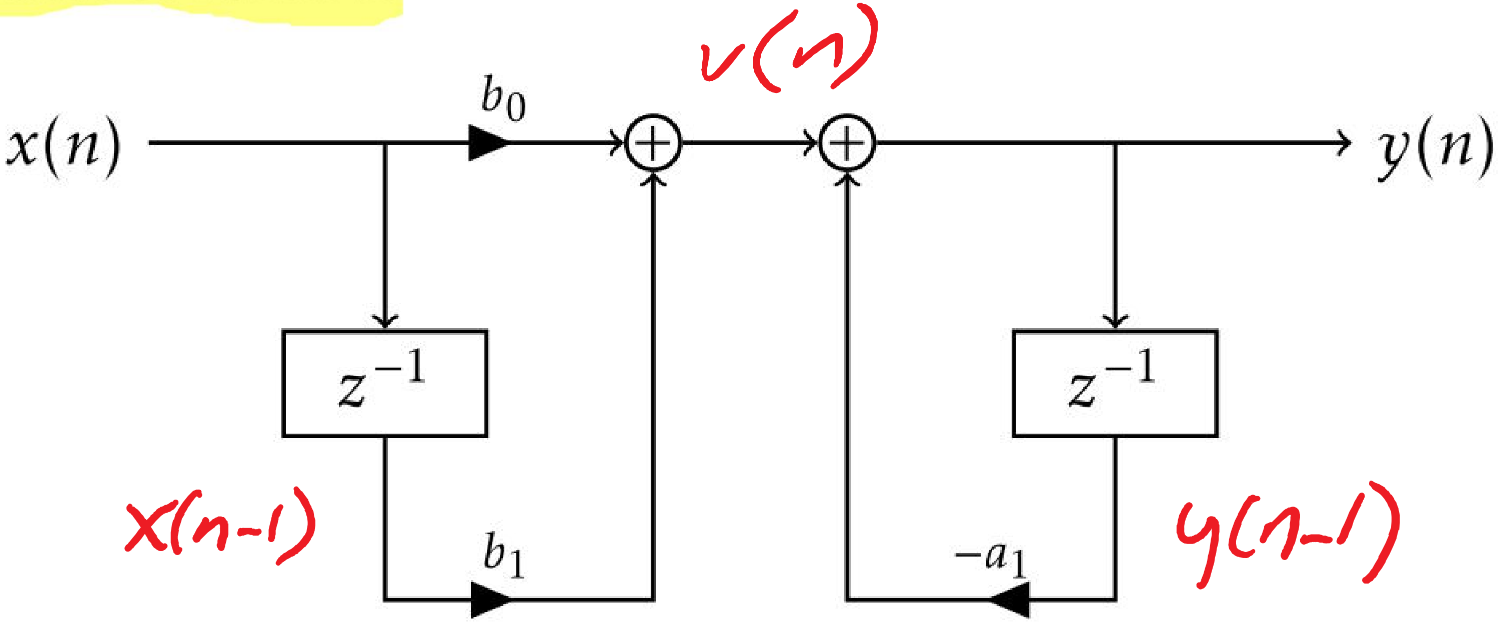
$$Y(z) = h(0)X(z) + h(1)z^{-1}X(z) + h(\overset{2}{\cancel{1}})z^{-2}X(z) + h(\overset{3}{\cancel{1}})z^{-3}X(z) = \underbrace{\sum_{k=0}^{3} h(k)z^{-k}}_{H(z)} X(z)$$

IIR filters

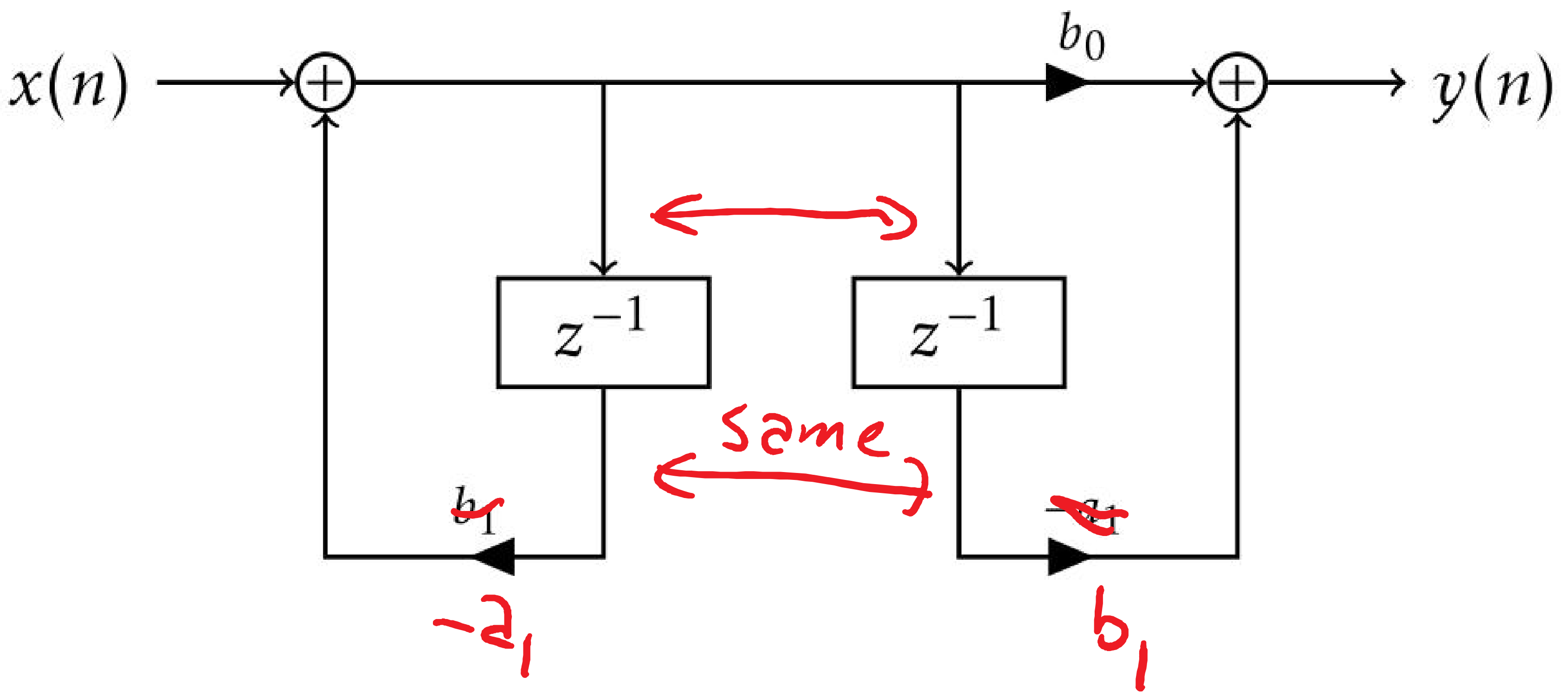
First order:

$$y(n)+a_1y(n-1)=\overbrace{b_0x(n)+b_1x(n-1)}^{v(n)} \Rightarrow y(n) = -a_1 y(n-1) + v(n)$$

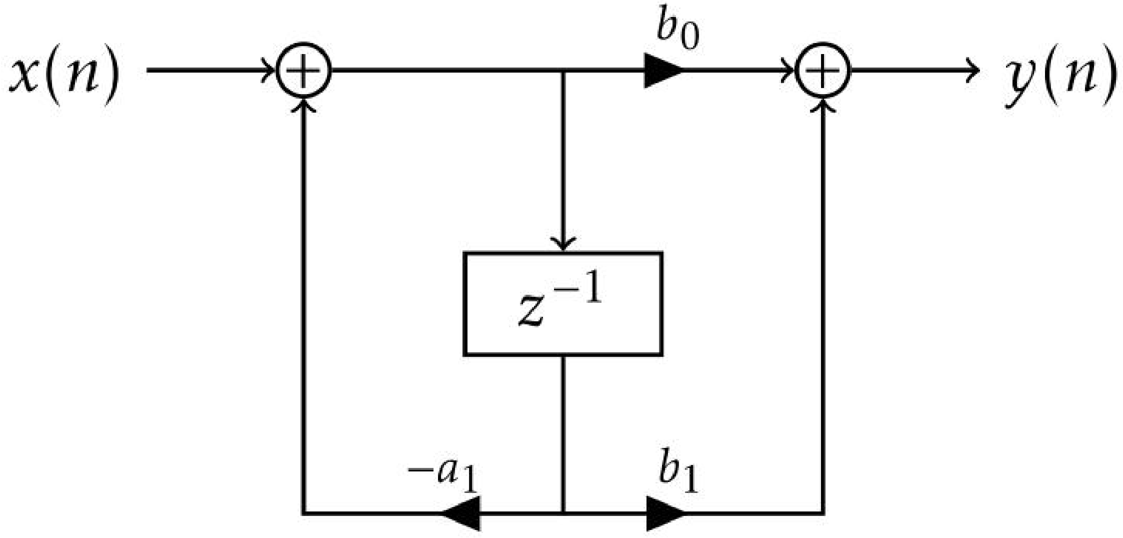
Can be drawn on direct form I.



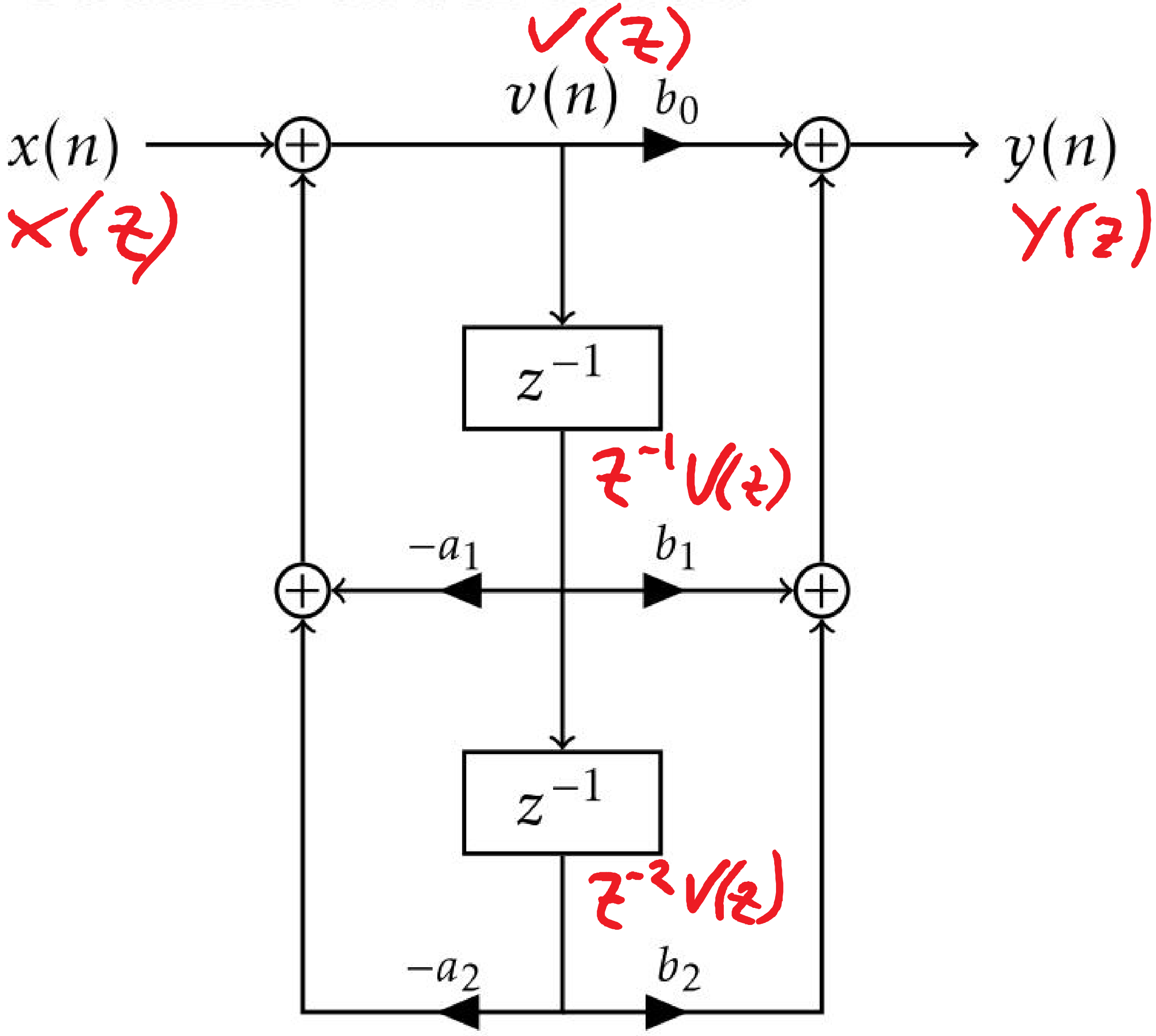
Since the system is linear we can change the order of the sub-systems.



The two delay lines can be merged to direct form II (normal form, canonical form).



Second order filter:



In the Z-domain we get;

$$V(z) = -z^{-1}a_1V(z) - z^{-2}a_2V(z) + X(z)$$

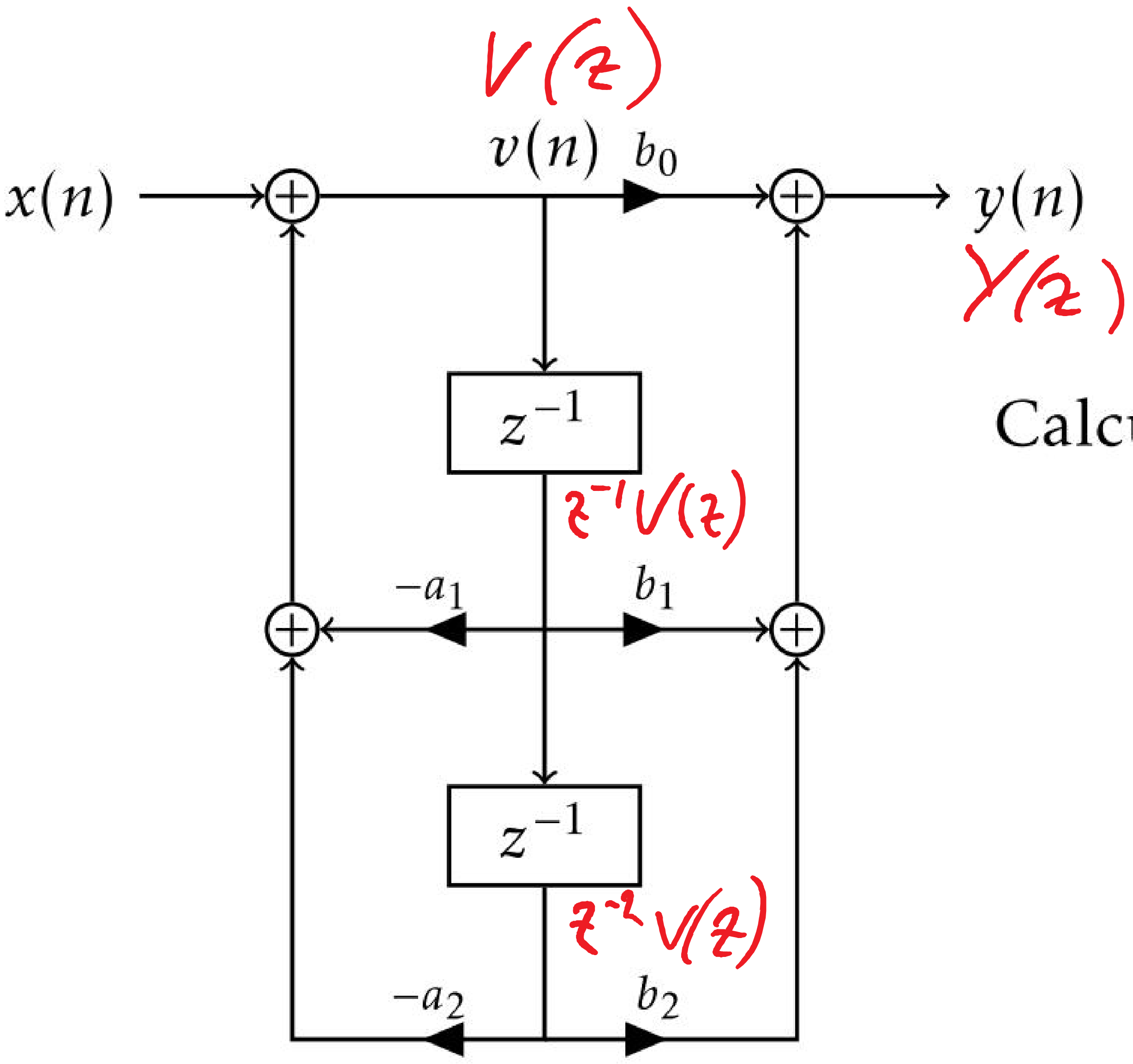
$$V(z) + z^{-1}a_1V(z) + z^{-2}a_2V(z) = X(z)$$

1+

$$V(z)\left(z^{-1}a_1 + z^{-2}a_2\right) = X(z)$$

$$V(z) = \frac{X(z)}{z^{-1}a_1 + z^{-2}a_2}$$

1+



We had from previous slide;

$$V(z) = \frac{X(z)}{1 + z^{-1}a_1 + z^{-2}a_2}$$

Calculate the output signal from $v(n)$.

$$Y(z) = b_0 V(z) + z^{-1} b_1 V(z) + z^{-2} b_2 V(z)$$

$$Y(z) = \left(b_0 + z^{-1} b_1 + z^{-2} b_2\right) V(z)$$

$$Y(z) = \frac{b_0 + z^{-1} b_1 + z^{-2} b_2}{\underbrace{1 + z^{-1}a_1 + z^{-2}a_2}_{H(z)}} \cdot X(z)$$

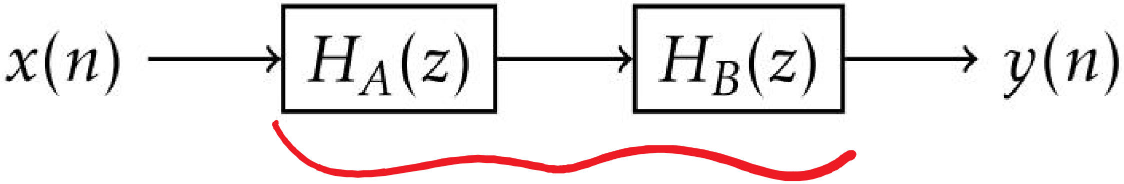
Parallel or cascade form

Ex: $$\boxed{H(z)=\frac{1}{1-\frac{1}{2}\cdot z^{-1}+\frac{1}{4}\cdot z^{-2}-\frac{1}{8}\cdot z^{-3}}}$$ [poles in $p_1 = 0.5$ och $p_{2,3} = \pm j0.5$]

$$=\underbrace{\frac{1}{1+\frac{1}{4}\cdot z^{-2}}}_{H_A}\cdot\underbrace{\frac{1}{1-\frac{1}{2}\cdot z^{-1}}}_{H_B}$$ [cascade coupling of $H_A(z)$ and $H_B(z)$]

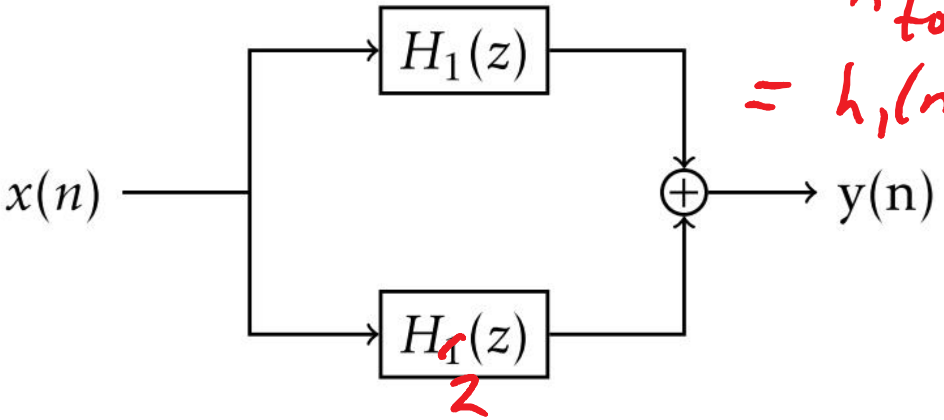
$$=\underbrace{\frac{\frac{1}{2}+\frac{1}{4}\cdot z^{-1}}{1+\frac{1}{4}\cdot z^{-2}}}_{H_1}+\underbrace{\frac{\frac{1}{2}}{1-\frac{1}{2}\cdot z^{-1}}}_{H_2}$$ [parallel coupling of $H_1(z)$ and $H_2(z)$]

Cascade (series) coupling:



$h_{tot}(n) = h_A(n) * h_B(n)$

Parallel coupling:

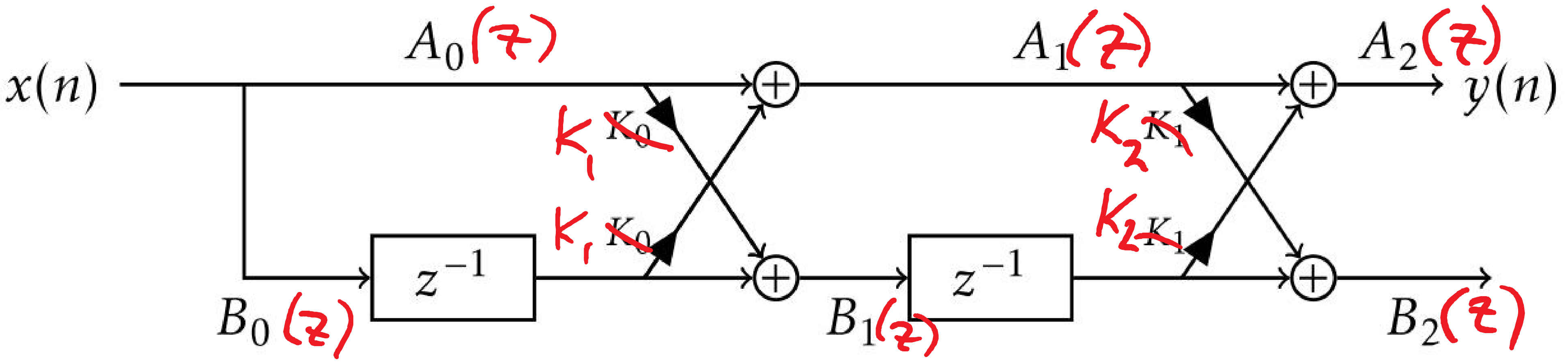


$h_{tot}(n) =$
$= h_1(n) + h_2(n)$

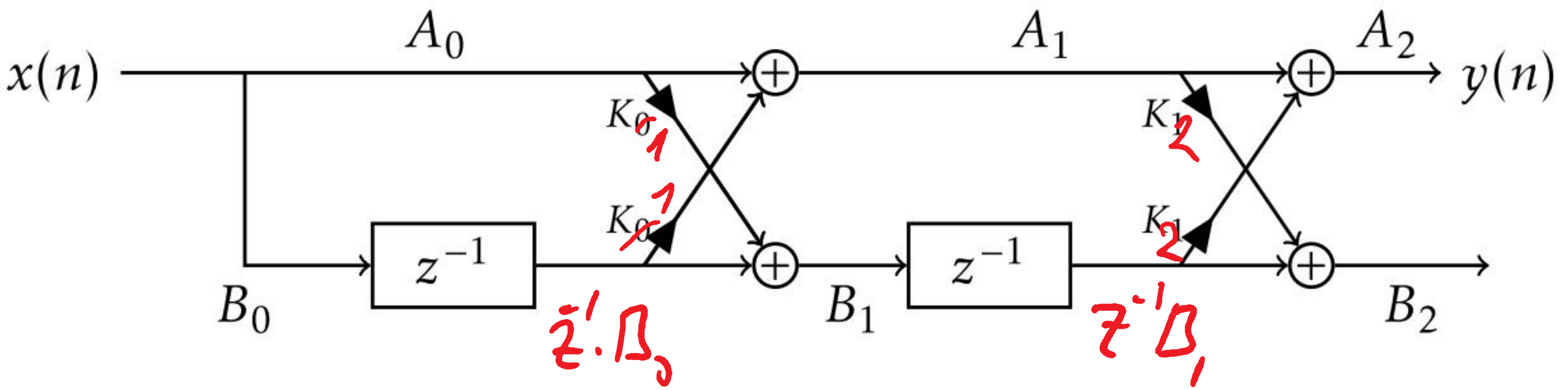
Lattice filter

A very common structure when modelling signals, especially speech signals.

Second order lattice FIR



If all $|K_i| < 1$ then all roots (zeros) are inside the unit circle.



Analysis of lattice FIR

Step 0:

$$A_0(z) = B_0(z) = 1$$

Step 1:

$A_0(z) = 1$

$$A_1(z) = 1 + K_1 z^{-1}$$

$$B_1(z) = K_1 + z^{-1}$$

OBS! From $A_2(z)$ we can find K_2

Step 2:

$$A_2(z) = A_1 + K_2 z^{-1} B_1(z) \quad = 1 + (K_1 + K_1 K_2) z^{-1} + K_2 z^{-2}$$

$$B_2(z) = K_2 A_1(z) + z^{-1} B_1(z) \quad = K_2 + (K_1 + K_1 K_2) z^{-1} + z^{-2}$$

$\boxed{B_2(z) \text{ can be obtained from } A_2(z) \text{ with the coefficients in reverse order.}}$

Step m:

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z)$$

$$B_m(z) = K_m A_{m-1}(z) + z^{-1} B_{m-1}(z)$$

In matrix form:

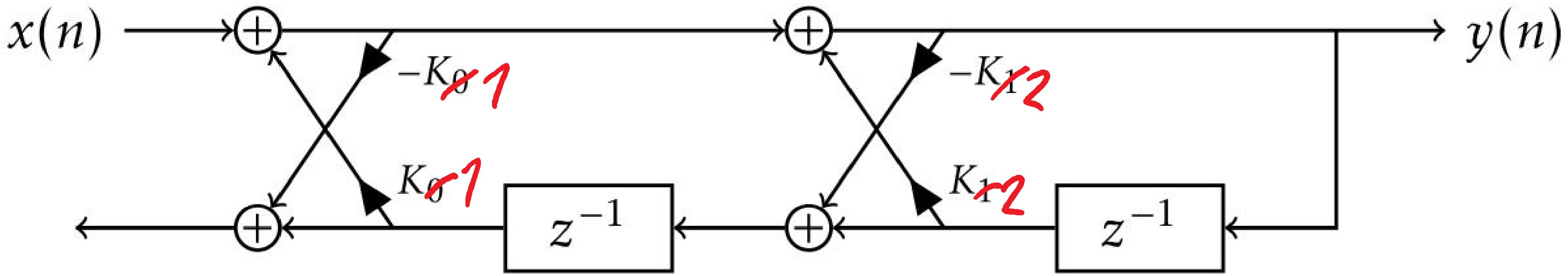
$$\begin{bmatrix} A_m(z) \\ B_m(z) \end{bmatrix} = \begin{bmatrix} 1 & z^{-1} K_m \\ K_m & z^{-1} \end{bmatrix} \cdot \begin{bmatrix} A_{m-1}(z) \\ B_{m-1}(z) \end{bmatrix}$$

Reverse:

$$A_{m-1}(z) = \frac{1}{1 - K_m^2} \cdot [A_m(z) - K_m B_m(z)]$$

Second order lattice all-pole IIR

All-pole filters have poles only (all zeros at the origin).



Compare with lattice FIR:

$$H(z) = \frac{1}{A(z)}$$

If all $|K_i| < 1$ then all roots (poles) are inside the unit circle.

This is used in GSM-codecs

Example

H(z) to Lattice

Given:

$$H(z) = 1 - z^{-1} + \frac{1}{2} \cdot z^{-2}$$

zeros

$$\left[\text{~~poles~~ in } z_{1,2} = \frac{1}{\sqrt{2}} \cdot e^{\pm j\pi \cdot \frac{1}{4}}\right]$$

Find: Calculate lattice FIR coefficients K_i.

Solution: Start with

$$A_2(z) = H(z) = 1 - z^{-1} + \frac{1}{2} \cdot z^{-2} \quad \Rightarrow \quad K_2 = \frac{1}{2}$$

$\Rightarrow K_2$

$$B_2(z) = \frac{1}{2} - z^{-1} + z^{-2}$$

Calculate in reverse:

$$A_1(z) = \frac{1}{1 - K_2^2} \cdot [A_2(z) - K_2 B_2(z)]$$

$$= \frac{1}{1 - \frac{1}{4}} \cdot \left[\underbrace{\left(1 - z^{-1} + \frac{1}{2} \cdot z^{-2}\right)}_{A_2(z)} - \overset{K_2}{\frac{1}{2}} \cdot \underbrace{\left(\frac{1}{2} - z^{-1} + z^{-2}\right)}_{B_2(z)}\right]$$

$$= 1 \underbrace{- \frac{2}{3}}_{\to K} \cdot z^{-1} \quad \Rightarrow \quad K_1 = -\frac{2}{3}$$

Finally, the answer is

$$K = \left\{ -\frac{2}{3} \quad \frac{1}{2} \right\} = \{K_1 \ \ K_2\}$$

Ex: Find H(z) from a given Lattice-structure

(Lattice to H(z))

Given: A Lattice-structure with parameters, $k_1 = \frac{1}{2}$, $k_2 = -\frac{1}{3}$, $k_3 = 1$

Find: The corresponding H(z)!

Solution:

Formula: $A_m(z) = A_{m-1}(z) + k_m z^{-1} B_{m-1}(z)$

Start with: $A_0(z) = B_0(z) = 1$

$m=1$, $\quad A_1(z) = A_0(z) + k_1 z^{-1} \cdot B_0(z) = 1 + \frac{1}{2} z^{-1} \qquad \left(k_1 = \frac{1}{2}\right)$

$\Rightarrow B_1(z) = \frac{1}{2} + z^{-1}$

$m=2$, $\quad A_2(z) = A_1(z) + k_2 z^{-1} B_1(z) = \underbrace{1 + \frac{1}{2} z^{-1}}_{A_1(z)} + k_2 z^{-1} \underbrace{\left(\frac{1}{2} + z^{-1}\right)}_{B_1(z)} = \left(k_2 = -\frac{1}{3}\right)$

$= 1 + \frac{1}{3} z^{-1} - \frac{1}{3} z^{-2} \quad \Rightarrow B_2(z) = -\frac{1}{3} + \frac{1}{3} z^{-1} + z^{-2}$

$m=3, \quad A_3(z) = A_2(z) + k_3 z^{-1} B_2(z) = 1 + z^{-3} \quad (k_3 = 1)$

$\Rightarrow$ Answer $H(z) = A_3(z) = \underline{\underline{1 + z^{-3}}}$

Algorithms Summary:

From lattice to system equation

Given: $K = \{\ K_1 \quad K_2 \quad \ldots \quad K_m\ \}$

Find: $H(z)$

Solution:

$$A_0(z) = B_0(z) = 1$$ ← Start with

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z)$$

$B_m(z) =$ coefficients in $A_m(z)$ in reverse order

iterate

$$H(z) = A_{M-1}(z)$$

Algorithms Summary:

From system equation to lattice

Given: $H(z)$

Find: $K = \{\, K_1 \quad K_2 \quad \ldots \quad K_m \,\}$

Solution:

$$A_{M-1}(z) = H(z)$$

start with

$$K_m = \text{coefficients for the terms } z^{-m}$$

Iterate

$$A_{m-1}(z) = \frac{1}{1-K_m^2} \cdot [A_m(z) - K_m B_m(z)]$$

$$B_{m-1}(z) = \text{coefficients in } A_m(z) \text{ in reverse order}$$