Lecture 2

Chapter 2

Convolution
Impulse response
Difference equations
Correlation functions

Let us look at a few simple examples that illustrates your previous familiarity of convolution!

Ex: Ordinary decimal multiplication: $101 \cdot 11 = 1111$, $(100 + 0 + 1) \cdot (10 + 1) = 1000 + 100 + 10 + 1$

Ex: Ordinary binary multiplication: $101 \cdot 11 = 1111$, $(4 + 0 + 1) \cdot (2 + 1) = 8 + 4 + 2 + 1 = 2^3 + 2^2 + 2^1 + 2^0$

$(5) \cdot (3) = (15)$ in decimal

Ex: Ordinary polynomial multiplication: $(x^2 + 1) \cdot (x + 1) = (1x^3 + 1x^2 + 1x + 1)$

$(1x^2 + 0x^1 + 1x^0) \cdot (1x + 1x^0)$

We will soon see that $(101) * (11) = (1111)$

Convolution operator

Assume we have a Linear and Time-Invariant system (LTI), here denoted by h(n) (it will soon be clear why we use this notation)

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

	Input signal	$\rightarrow$	Output signal
	$x(n)$	$\rightarrow$	$y(n)$
	$\delta(n)$	$\rightarrow$	$h(n)$
Time-Invariance =>	$\delta(n-k)$	$\rightarrow$	$h(n-k)$
Linearity =>	$x(k)\delta(n-k)$	$\rightarrow$	$x(k)h(n-k)$
Linearity =>	$x(n) = \sum_k x(k)\delta(n-k)$	$\rightarrow$	$\sum_k x(k)h(n-k)$

$$\boxed{y(n) = \sum_k x(k)h(n-k) = \sum_k h(k)x(n-k) = h(n) * x(n)}$$

From previous lecture!

Using the impulse we can write

Ex

$$x(n) = \{ \underline{1} \quad 4 \quad 1 \} = 1\cdot\delta(n) + 4\cdot\delta(n-1) + 1\cdot\delta(n-2) = \boxed{\sum_k x(k)\delta(n-k)}$$

n=0

... 0 0 0 1 0 0 0 ...

... 0 0 0 0 4 0 0 0 ...

+ ... 0 0 0 0 0 1 0 0 0 ...

0 0 0 0 0 0 1 4 1 0 0 0 0

n=0

Example of convolution

Given: Input signal $x(n)$ and impulse response $h(n)$.

$$x(n) = \left\{ \underline{2} \quad 4 \quad 6 \quad 4 \quad 2 \right\}$$

$$h(n) = \left\{ \underline{3} \quad 2 \quad 1 \right\}$$

Find: Output signal $y(n)$.

$$y(n) = \sum_k h(n-k)x(k) = \sum_k h(k)x(n-k)$$

$$= \underbrace{h(0)x(n)}_{k=0} + \underbrace{h(1)x(n-1)}_{k=1} + \underbrace{h(2)x(n-2)}_{k=2}$$

$$= 3x(n) + 2x(n-1) + x(n-2)$$

$$\begin{array}{r} 3\cdot(2\ 4\ 6\ 4\ 2) \\ 2\cdot(0\ 2\ 4\ 6\ 4\ 2) \\ +\ 1\cdot(0\ 0\ 2\ 4\ 6\ 4\ 2) \\ \hline \underline{6}\ 16\ 28\ 28\ 20\ 8\ 2 \end{array}$$

Solution: We solve the convolution graphically with the following visual procedure

For $n = 0$:

	$h(0-k)$	1	2	$\underline{3}$					
	$x(k)$			$\underline{2}$	4	6	4	2	
$\sum_k$	$h(0-k)x(k)$			$\underline{6}$					$\sum = \underline{6} = y(0)$

For $n = 1$:

	$h(1-k)$	1	2	$\underline{3}$		$\rightarrow$ 1 step			
	$x(k)$		$\underline{2}$	4	6	4	2		
$\sum_k$	$h(1-k)x(k)$		4	12					$\sum = 16 = y(1)$

For $n = 2$:

$h(2-k)$	1	2	$\underline{3}$			$\rightarrow$ 2 steps
$x(k)$	$\underline{2}$	4	6	4	2	
$\sum_k h(2-k)x(k)$	2	8	18			$\sum = 28 = y(2)$

We continue to do the same process for n=3,4,5,6, and we get

$$y(n) = \{\ \underline{6}\quad 16\quad 28\quad 28\quad 20\quad 8\quad 2\ \}$$

We can formalize the procedure by using a Convolution Table =>

We had;

$$x(n) = \{\ \underline{2}\quad 4\quad 6\quad 4\quad 2\ \}$$

$$h(n) = \{\ \underline{3}\quad 2\quad 1\ \}$$

$h(n)$ \ $x(n)$	$\underline{2}$	4	6	4	2
$\underline{3}$	$\underline{6}$	12	18	12	6
2	4	8	12	8	4
1	2	4	6	4	2

$\underline{6}$ 16 28 28 20 8 2

Properties of convolution (page 81)

Commutativity

$$x_1(n) * x_2(n) = x_2(n) * x_1(n)$$

Associativity

$$x_1(n) * [x_2(n) * x_3(n)] = [x_1(n) * x_2(n)] * x_3(n)$$

Distributivity

$$x_1(n) * [x_2(n) + x_3(n)] = x_1(n) * x_2(n) + x_1(n) * x_3(n)$$

Input-output

$$y(n) = x(n) * h(n)$$

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

Cascade or Serial coupling

$$y(n) = x(n) * h_1(n) * h_2(n)$$

$$h(n) = h_1(n) * h_2(n)$$

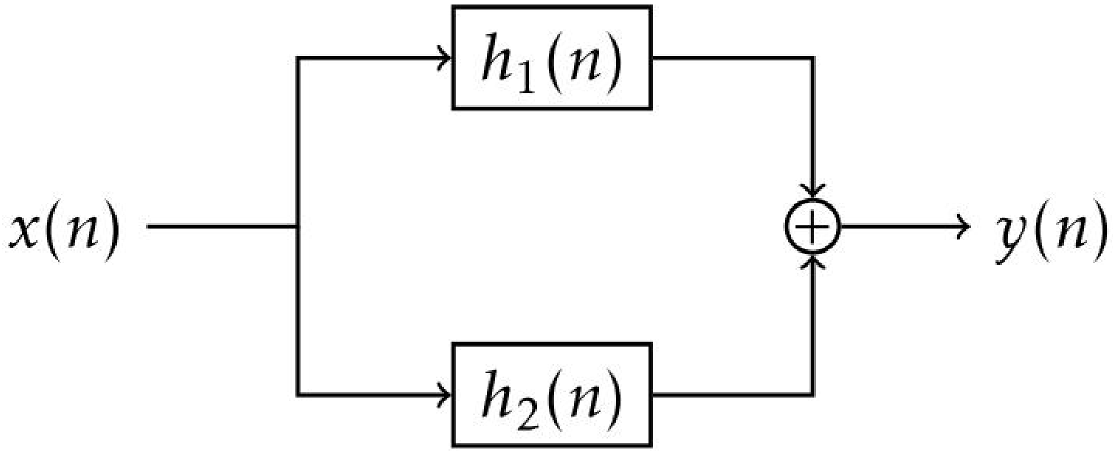
$$x(n) \longrightarrow \boxed{h_1(n)} \longrightarrow \boxed{h_2(n)} \longrightarrow y(n)$$

$$\Leftrightarrow \qquad x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

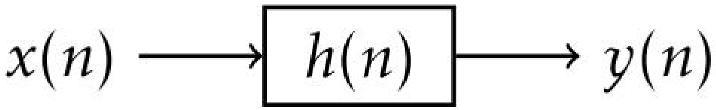
Parallel coupling

$$y(n) = [x(n) * h_1(n)] + [x(n) * h_2(n)] = x(n) * [h_1(n) + h_2(n)]$$

$$h(n) = h_1(n) + h_2(n)$$



$\Leftrightarrow$



Stability (sid 85)

A system is BIBO-stable (bounded input-bounded output) if

$$|x(n)| \leq M_x \quad \Rightarrow \quad |y(n)| \leq M_y$$

or equivalently

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)x(n-k) \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)|$$

$$\leq M_x \cdot \sum_{k=-\infty}^{\infty} |h(k)|$$

The system is therefore stable if

$$\boxed{\sum_{k=-\infty}^{\infty} |h(k)| \leq \infty}$$

Ex: $\left|\overbrace{-3\cdot 2}^{-6} + \overbrace{3\cdot 2}^{6}\right| = 0$

$\leq \underbrace{|-3|\cdot|2|}_{6} + \underbrace{|3|\cdot|2|}_{6} = 12$

Difference equations (page 93–95)

General:

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{N} b_k x(n-k)$$

Example

The FIR-filter

$$y(n) = 0.5x(n) + 0.25x(n-1) + 0.15x(n-2) \quad \Rightarrow \quad h(n) = \{ \underline{0.5} \quad 0.25 \quad 0.15 \}$$

Since, if the input $x(n)=\delta(n)$ => $y(n) = h(n) = 0.5\,\delta(n) + 0.25\,\delta(n-1) + 0.15\,\delta(n-2) = \{\underline{0.5}\ 0.25\ 0.15\}$

Example

A first order IIR-filter:

$$y(n) = 0.5y(n-1) + 2x(n)$$

A second order IIR-filter:

$$y(n) = 0.5y(n-1) + 0.5y(n-2) + x(n)$$

We will use the Z-transform to solve a recursive difference equation (IIR) (Chapter 3)

Example

Given:

$$h(n) = \left(\frac{1}{2}\right)^n \cdot u(n)$$

$$x(n) = u(n)$$

Find:

$$y(n) = h(n) * x(n)$$

Solution: Convolution gives

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} \underbrace{\overbrace{\left(\frac{1}{2}\right)^k u(k)}^{=0 \text{ for } k<0}}_{h(k)} \cdot \underbrace{\overbrace{u(n-k)}^{=0 \text{ for } n-k<0}}_{x(n-k)}$$

$\Leftrightarrow k > n$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k$$

$$= \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 2-\left(\frac{1}{2}\right)^n \qquad n \geq 0$$

The solution is therefore

$$\boxed{y(n) = \left[2-\left(\frac{1}{2}\right)^n\right] \cdot u(n)}$$

Correlation functions (sid 118)

How similar are two signals?

Auto correlation function

$$r_{xx}(k) = \sum_{n=-\infty}^{\infty} x(n)x(n-k) = x(k) * x(-k)$$

Cross correlation function

$$r_{yx}(k) = \sum_{n=-\infty}^{\infty} y(n)x(n-k) = y(k) * x(-k)$$

Cross correlation for input and output signals

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

The auto correlation for the input signal:

$$r_{xx}(k) = x(k) * x(-k)$$

The cross correlation between the input signal and the output signal:

$$r_{yx}(k) = y(k) * x(-k)$$

$$= h(k) * \underbrace{x(k) * x(-k)}$$

$$= h(k) * r_{xx}(k)$$

The auto correlation for the output signal:

$$r_{yy}(k) = y(k) * y(-k)$$

$$= \underbrace{h(k) * x(k)}_{y(k)} * \underbrace{h(-k) * x(-k)}_{y(-k)}$$

$$= r_{hh}(k) * r_{xx}(k)$$

Ex) A simple radar System