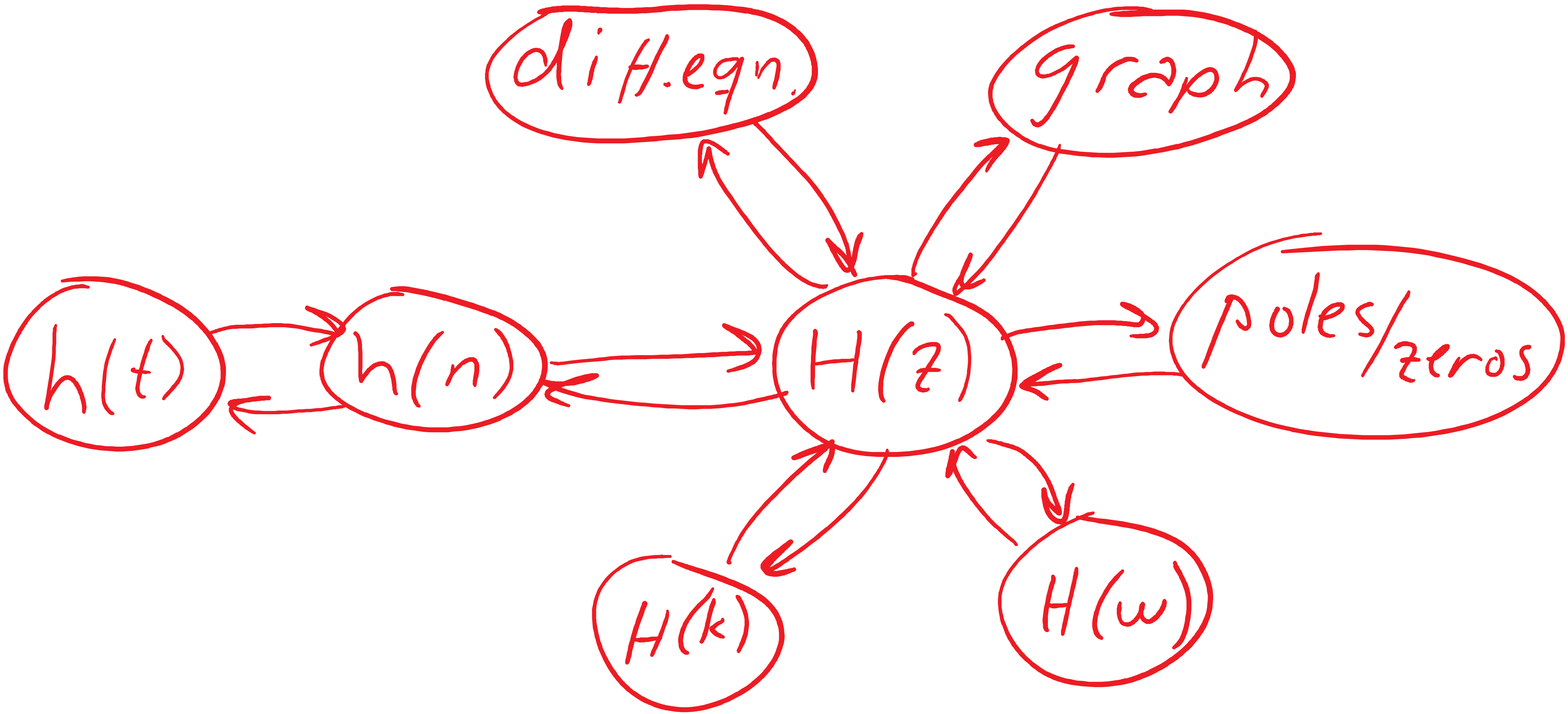
Lecture 3

Digital Signal Processing

Chapter 3

z-transforms


diff.eqn.
graph
h(t)
h(n)
H(z)
poles/zeros
H(k)
H(ω)

z-transforms

We define the z-transform

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

z-transforms

We define the z-transform

$$\boxed{H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}}$$

$H(z)$ is a complex valued function

$$H(z)\colon\ \mathbb{C} :=> \mathbb{C}$$

z-transforms

We define the z-transform

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} = \sum_{n=0}^{\infty} h(n) z^{-n}$$

if $h(n)$ caasual

Example

$$\boxed{H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}}$$

Function	$\Leftrightarrow$	z-transform
$h(n)$	$\Leftrightarrow$	$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots$
$\delta(n) = \{\ \underline{1}\ \ 0\ \ \ldots\ \}$	$\Leftrightarrow$	1
$\delta(n-k)$	$\Leftrightarrow$	$z^{-k} = \sum_{n=k}^{k} z^{-n}$
$h(n-k)$	$\Leftrightarrow$	$z^{-k} H(z)$
$h_1(n) = \{\ \underline{3}\ \ 2\ \ 1\ \}$	$\Leftrightarrow$	$H_1(z) = 3 + 2z^{-1} + z^{-2}$
$h_2(n) = \{\ \underline{0}\ \ 3\ \ 2\ \ 1\ \}$	$\Leftrightarrow$	$H_2(z) = 0 + 3z^{-1} + 2z^{-2} + z^{-3} = z^{-1} \underbrace{\left(3 + 2z^{-1} + z^{-2}\right)}_{H_1(z)}$

Proof for the time delay.

$$y(n) = x(n-1) \quad \Leftrightarrow \quad Y(z) = \sum_n y(n) z^{-n}$$

$$= \sum_n x(n-1) \underbrace{z^{-n} \cdot z^{1}} \cdot z^{-1}$$

$$= z^{-1} \sum_n x(n-1) z^{-(n-1)}$$

$$= z^{-1} \underbrace{\sum_m x(m) z^{-m}}$$

$$= z^{-1} X(z)$$

Example

An IIR-system and its z-transform.

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n$$

$$= \frac{1-(z^{-1})^{\infty+1}}{1-z^{-1}}$$

$$= \frac{1}{1-z^{-1}} \qquad \text{if } |z| > 1 \text{ (ROC)}$$

If $|z^{-1}| < 1 \Longrightarrow |z| > 1$

ROC means *region of convergence:* for which z the sum converges.

Example

$$h(n) = a^n \cdot u(n) \quad \Leftrightarrow \quad H(z) = \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(a \cdot z^{-1}\right)^n$$

$$= \frac{1 - (a \cdot z^{-1})^{\infty+1}}{1 - z^{-1}}$$

$$= \frac{1}{1 - a \cdot z^{-1}} \qquad \text{if } |z| > |a| \text{(ROC)}$$

if $|a \cdot z^{-1}| < 1 \Rightarrow |z| > |a|$

ROC means *region of convergence:* for which z the sum converges.

Example of z-transform of non-causal signal (page 154)

Given:

$$x(n) = \left(\frac{1}{2}\right)^{|n|} \qquad \text{for all } n$$

Find: The z-transform $X(z)$ of $x(n)$.

Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n|} \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^{-n} \cdot z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \cdot z^{-n} - 1$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \cdot z^{n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \cdot z^{-n} - 1$$

$$= \frac{1 - \left(\frac{1}{2} \cdot z\right)^{\infty+1}}{1 - \frac{1}{2} \cdot z} + \frac{1 - \left(\frac{1}{2} \cdot z^{-1}\right)^{\infty+1}}{1 - \frac{1}{2} \cdot z^{-1}} - 1$$

$$\left\{\text{if } |z| < 2 \text{ and } |z| > \frac{1}{2} \text{ (ROC)}\right\} = \frac{1}{1 - \frac{1}{2} \cdot z} + \frac{1}{1 - \frac{1}{2} \cdot z^{-1}} - 1 \qquad = \frac{1 - \left(\frac{1}{2}\right)^2}{\left(1 - \frac{1}{2} \cdot z\right)\left(1 - \frac{1}{2} \cdot z^{-1}\right)}$$

Convolution becomes multiplication

$$y(n) = h(n) * x(n) \quad \Leftrightarrow \quad Y(z) = H(z)X(z)$$

Proof:

$$\begin{aligned}
Y(z) &= \sum_n y(n) z^{-n} \\
&= \overbrace{\sum_n \sum_k h(k)x(n-k)}^{y(n)} z^{-n} \cdot z^{k} \cdot z^{-k} \\
&= \sum_n \sum_k h(k)x(n-k) z^{-(n-k)} z^{-k} \\
&= \underbrace{\sum_k h(k) z^{-k}}_{H(z)} \cdot \underbrace{\sum_n x(n-k) z^{-(n-k)}}_{X(z)} \\
&= H(z)X(z)
\end{aligned}$$

Cascade or Serial coupling of systems

Cascading two systems gives:

$$x(n) \longrightarrow \boxed{h_1(n)} \xrightarrow{y_1(n)} \boxed{h_2(n)} \longrightarrow y(n)$$

$$\Leftrightarrow$$

$$x(n) \longrightarrow \underbrace{\boxed{h(n)}}_{h_{tot}(n) = h_1(n) * h_2(n)} \longrightarrow y(n)$$

$$H_{tot}(z) = H_1(z)H_2(z)$$

Example of solving a difference equation using the Z-transform

Determine the balance of a bank account with interest.

Given: Deposit is 100 every year with 5% interest.

$$x(n) = 100 \cdot u(n)$$

$$y(n) = \text{balance at year } n$$

Find: Balance after 1, 2, 5 and 20 years.

Solution: The current balance is the balance from last year plus 5% interest and the deposit for the current year.

$$y(n) = 1.05y(n-1) + x(n) \tag{51}$$

$$x(n) = 100 \cdot u(n)$$

Apply the z-transform.

$$Y(z) = 1.05 \cdot z^{-1} Y(z) + X(z) \Leftrightarrow Y(z) - 1.05 z^{-1} Y(z) = X(z)$$

$$X(z) = 100 \cdot \frac{1}{1 - z^{-1}} \quad \Leftrightarrow Y(z)(1 - 1.05 z^{-1}) = X(z)$$

Solve for $Y(z)$.

$$Y(z) = \underbrace{\frac{1}{1 - 1.05 \cdot z^{-1}}}^{H(z)} \cdot X(z)$$

$$= \frac{1}{1 - 1.05 \cdot z^{-1}} \cdot \frac{100}{1 - z^{-1}} = \{\text{partial fraction}\} =$$

$$= 100 \cdot \left(\frac{21}{1 - 1.05 \cdot z^{-1}} - \frac{20}{1 - z^{-1}} \right)$$

Apply inverse-transforms to obtain the time-sequence.

$$\boxed{y(n) = 100 \cdot (21 \cdot 1.05^n - 20)\, u(n)}$$

$$= \left\{ \begin{array}{ccccccccc} 100 & 205 & 315 & \ldots & 680 & \ldots & 3572 & \ldots & 10^{45} \\ n=0 & 1 & 2 & & 5 & & 20 & & 2060 \end{array} \right\}$$

Given a second order difference equation:

$$y(n) - 1.27y(n-1) + 0.81y(n-2) = x(n-1) - x(n-2)$$

z-transform each term in the equation

$$Y(z) - 1.27z^{-1}Y(z) + 0.81z^{-2}Y(z) = z^{-1}X(z) - z^{-2}X(z)$$

Solve for $Y(z)$.

$$Y(z) = \underbrace{\frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}}}_{H(z)} \cdot X(z) = H(z)X(z)$$

Inverse Transform to get the solution $y(n)$?

A short example as reminder of (distinct-pole) partial fraction:

$$\boxed{H(z) = \frac{1}{(z-a)(z-b)} = \frac{A}{(z-a)} + \frac{B}{(z-b)}}$$

In order to directly find A, we multiply all sides by the factor (z-a)

$$\Leftrightarrow$$

$$H(z)(z-a) = \frac{1\cancel{(z-a)}}{\cancel{(z-a)}(z-b)} = \frac{A\cancel{(z-a)}}{\cancel{(z-a)}} + \frac{B(z-a)}{z-b}$$

$$\Leftrightarrow$$

$$H(z)(z-a) = \frac{1}{(z-b)} = A + \cancel{\frac{B(z-a)}{z-b}} \quad \text{if } z=a$$

Now, by putting z=a we will force the rightmost term to zero (since a is a root to (z-a)), which will give the so called “Hands-on Approach”;

$$A = H(z)(z-a)\Big|_{z=a} = \frac{1}{(z-b)}\Big|_{z=a} = \frac{1}{(a-b)}$$

In the same way we can solve for B;

$$B = H(z)(z-b)\Big|_{z=b} = \frac{1}{(z-a)}\Big|_{z=b} = \frac{1}{(b-a)}$$

Example: Fibonacci sequence (page 210)

$$y(n) = \{ \underline{1} \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad \ldots \}$$

$$\Leftrightarrow$$

$$y(n) = y(n-1) + y(n-2) \qquad \text{where } y(0) = 1 \text{ and } y(1) = 1$$

find a closed form equation for this sequence

Rewrite the difference equation (by including an impulse response as input which replaces the initial conditions, y(1) and y(2))

$$y(n) = y(n-1) + y(n-2) + \delta(n)$$, $y(-1) = y(-2) = 0$

Apply the z-transform.

$$Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + 1$$

$$= \frac{1}{1 - z^{-1} - z^{-2}} = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}}$$

$$p_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2}(1 \pm \sqrt{5})$$

where

$$p_1 = \frac{1}{2}\left(1+\sqrt{5}\right) \qquad A_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}$$

$$p_2 = \frac{1}{2}\left(1-\sqrt{5}\right) \qquad A_2 = -\frac{1-\sqrt{5}}{2\sqrt{5}}$$

inverse z-transform $\dfrac{A_1}{1-p_1 z^{-1}} + \dfrac{A_2}{1-p_2 z^{-1}}$

$$\boxed{y(n) = A_1 p_1^n + A_2 p_2^n \qquad \text{for } n \geq 0}$$

Inverse z-transform

Three methods, where the third C) is the one we will use;

A)

Definition (page 181)

The definition of the inverse z-transform is

$$y(n) = \frac{1}{2\pi \mathrm{j}} \cdot \oint_C Y(z) z^{n-1} \, \mathrm{d}z \tag{52}$$

where C is a closed path around the origin and inside the ROC. Solved using residual calculus.

B) **Polynomial division (page 183)**

$$Y(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

$$= 1 + \frac{3}{2} \cdot z^{-1} + \frac{7}{4} \cdot z^{-2} + \frac{15}{8} \cdot z^{-3} + \cdots$$

$$y(n) = \left\{ \underline{1} \quad \tfrac{3}{2} \quad \tfrac{7}{4} \quad \tfrac{15}{8} \quad \ldots \right\}$$

Ex

$$1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}\ \Big)\ 1 \qquad\qquad 1+\frac{3}{2}z^{-1}+\frac{7}{4}z^{-2}+\frac{15}{8}z^{-3}\ \cdots\cdots$$

$$-\left(1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}\right)$$

$$\cancel{\frac{3}{2}}z^{-1}-\frac{1}{2}z^{-2}$$

$$-\left(\cancel{\frac{3}{2}}z^{-1}-\frac{9}{4}z^{-2}+\frac{3}{4}z^{-3}\right)$$

$$\cancel{\frac{7}{4}}z^{-2}-\frac{3}{4}z^{-3}$$

$$-\left(\cancel{\frac{7}{4}}z^{-2}-\frac{21}{8}z^{-3}+\frac{7}{8}z^{-4}\right)$$

$$\frac{15}{8}z^{-3}-\frac{7}{8}z^{-4}$$

$$\vdots$$

C) Use table of known transforms

First order

$$Y(z) = \frac{1}{1 - 0.9z^{-1}}$$

$$y(n) = 0.9^n \cdot u(n)$$

Second order with real poles by partial fraction expansion

$$Y(z) = \frac{1}{1 - \frac{3}{2} \cdot z^{-1} + \frac{1}{2} \cdot z^{-2}} = \frac{2}{1 - z^{-1}} - \frac{1}{1 - \frac{1}{2} \cdot z^{-1}}$$

$$y(n) = 2u(n) - \left(\frac{1}{2}\right)^n \cdot u(n) \quad \rightarrow 2 \quad n \rightarrow \infty$$

Second order with complex poles by table of formulas

$$Y(z) = \frac{\frac{1}{2} \cdot \sin\left(\frac{\pi}{4}\right) \cdot z^{-1}}{1 - 2\cos\left(\frac{\pi}{4}\right) \cdot z^{-1} + \frac{1}{4} \cdot z^{-2}}$$

$$y(n) = \left(\frac{1}{2}\right)^n \cdot \sin\left(\frac{\pi}{4} n\right) u(n)$$

Solving difference equations with initial values

We define the *one sided z*-transform as

$$\boxed{Y^{+}(z) = \sum_{n=0}^{\infty} y(n) z^{-n}} \qquad \text{even if } y(n) \neq 0 \text{ for } n < 0$$

Using this definition the z-transform of time shift is different

Assuming $y_0(n) = y(n-1)$

$$Y_0^+(z) = \sum_{n=0}^{\infty} y_0(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} y(n-1) z^{-n} \cdot z \cdot z^{-1}$$

$$= y(-1) + \underbrace{\sum_{n=1}^{\infty} y(n-1) z^{-(n-1)}} z^{-1}$$

$$= z^{-1} Y^+(z) + y(-1)$$

Common:

$$x(n) \quad \Leftrightarrow \quad X^{+}(z)$$

$$x(n-k) \quad \Leftrightarrow \quad z^{-k} \cdot \left[X^{+}(z) + \sum_{n=\cancel{0}\,1}^{k} x(-n) z^{n} \right]$$

$$= \left[\underbrace{x(-k)}_{n=k} + x(-k+1) z^{-1} + \cdots + \underbrace{x(-1) z^{-k+1}}_{n=1} + z^{-k} X^{+}(z) \right]$$

Example

Given: $y(n) = ay(n-1) + x(n)$ with a given initial value for $y(-1)$.

Find: $y(n)$ for $n \geq 0$.

Solution:

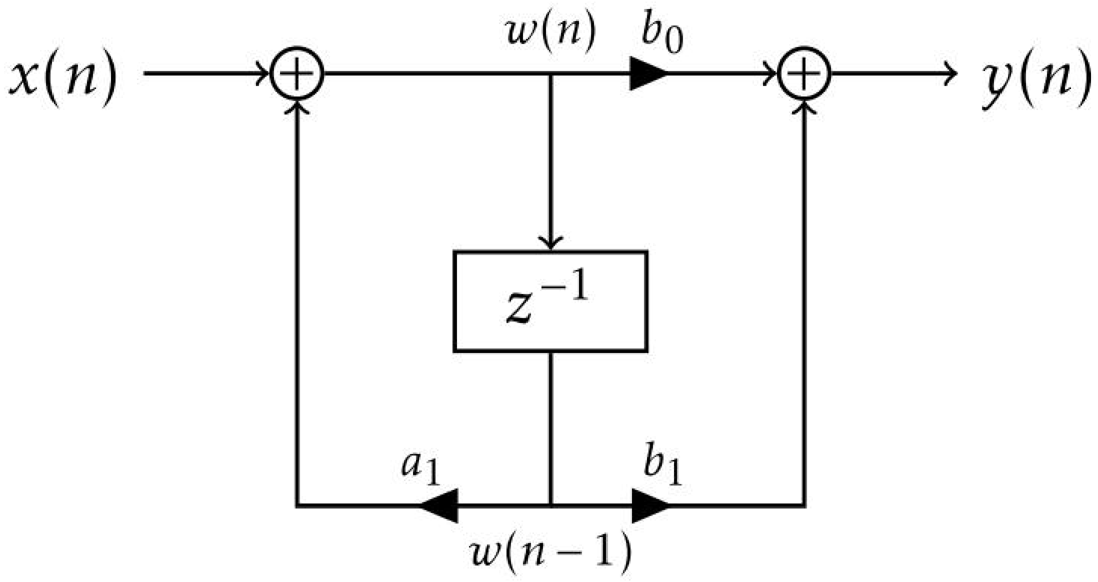
$$Y^+(z) = a \cdot \overbrace{\left[z^{-1}Y^+(z) + \underline{y(-1)}\right]}^{y(n-1)} + X(z)$$

$$= \frac{1}{1-az^{-1}} \cdot X(z) + \frac{a}{1-az^{-1}} \cdot y(-1)$$

Inverse transform with the given values of $X(z)$ and $y(-1)$.

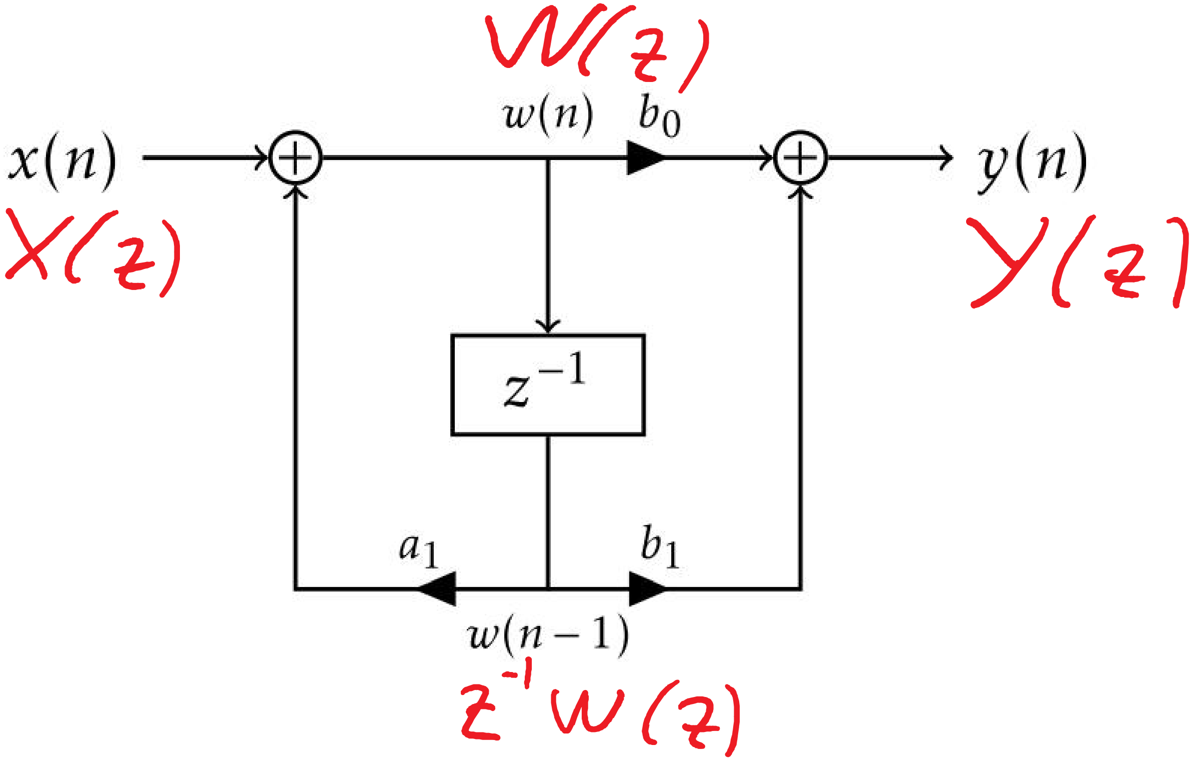
Direct form II (page 265)

Given: System drawn on the form:



Find: The connection between $x(n)$ and $y(n)$.

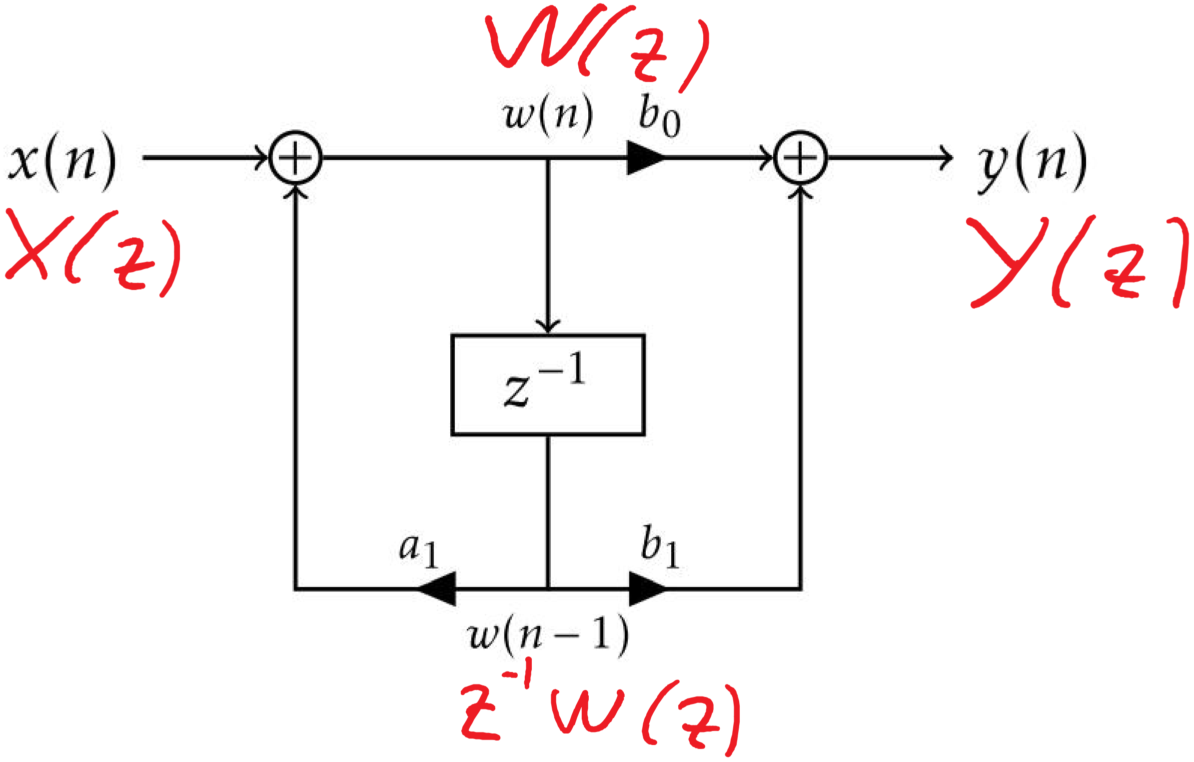
Step 1:



$$W(z) = a_1 z^{-1} W(z) + X(z)$$

$$= \frac{1}{1 - a_1 z^{-1}} \cdot X(z)$$

Step 2:



$$Y(z) = b_1 z^{-1} W(z) + b_0 W(z) = (b_0 + b_1 z^{-1}) \cdot W(z)$$

$$= \boxed{\frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}} X(z)$$

W(z)

$$= H(z)X(z)$$