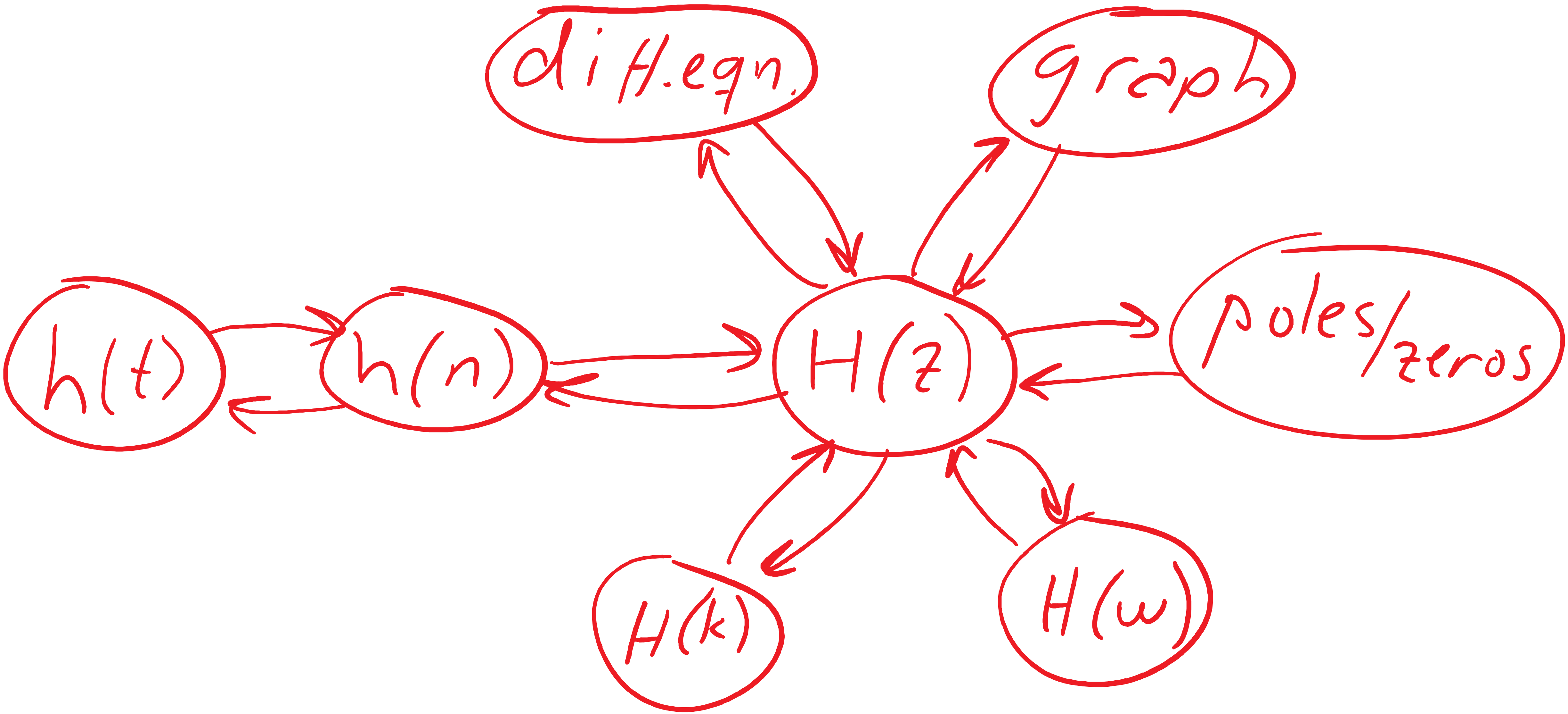
Lecture 4
Digital Signal Processing

Chapter 3

z-transforms


diff.eqn.
graph
h(t)
h(n)
H(z)
poles/zeros
H(k)
H(ω)

z-transform

We defined the z-transform of the impulse response $h(n)$ as

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} = \sum_{n=0}^{\infty} h(n) z^{-n}$$

where $h(n) = 0$ for $n < 0$ and where $z = r\mathrm{e}^{\mathrm{j}\omega}$.

Poles and zeros

Ex: Find the System function H(z) from a given difference equation

$$y(n) - 1.27y(n-1) + 0.81y(n-2) = x(n-1) - x(n-2)$$

$$Y(z) - 1.27z^{-1}Y(z) + 0.81z^{-2}Y(z) = z^{-1}X(z) - z^{-2}X(z)$$

$$Y(z) = \underbrace{\frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}}}_{H(z)} \cdot X(z) = H(z)X(z)$$

$$H(z) = \frac{B(z)}{A(z)}$$

$$= \frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}} \cdot \frac{z^2}{z^2}$$

$$= \frac{z - 1}{z^2 - 1.27z + 0.81}$$

Finding the roots to the denominator polynomial

$$z^2 - 1.27z + 0.81 = 0$$

$\Rightarrow$ poles

Finding the roots to the numerator polynomial

$$z - 1 = 0 \quad \rightarrow \quad z = 1$$

$\Rightarrow$ zeros

Poles:

$$z = \frac{1.27}{2} \pm \sqrt{\left(\frac{1.27}{2}\right)^2 - 0.81}$$

$$= 0.64 \pm j0.64$$

$$= 0.9e^{\pm j\frac{\pi}{4}}$$

Zeros: $z - 1 = 0 \quad \rightarrow \quad z = 1$

The factoring of $H(z)$ can now be written as

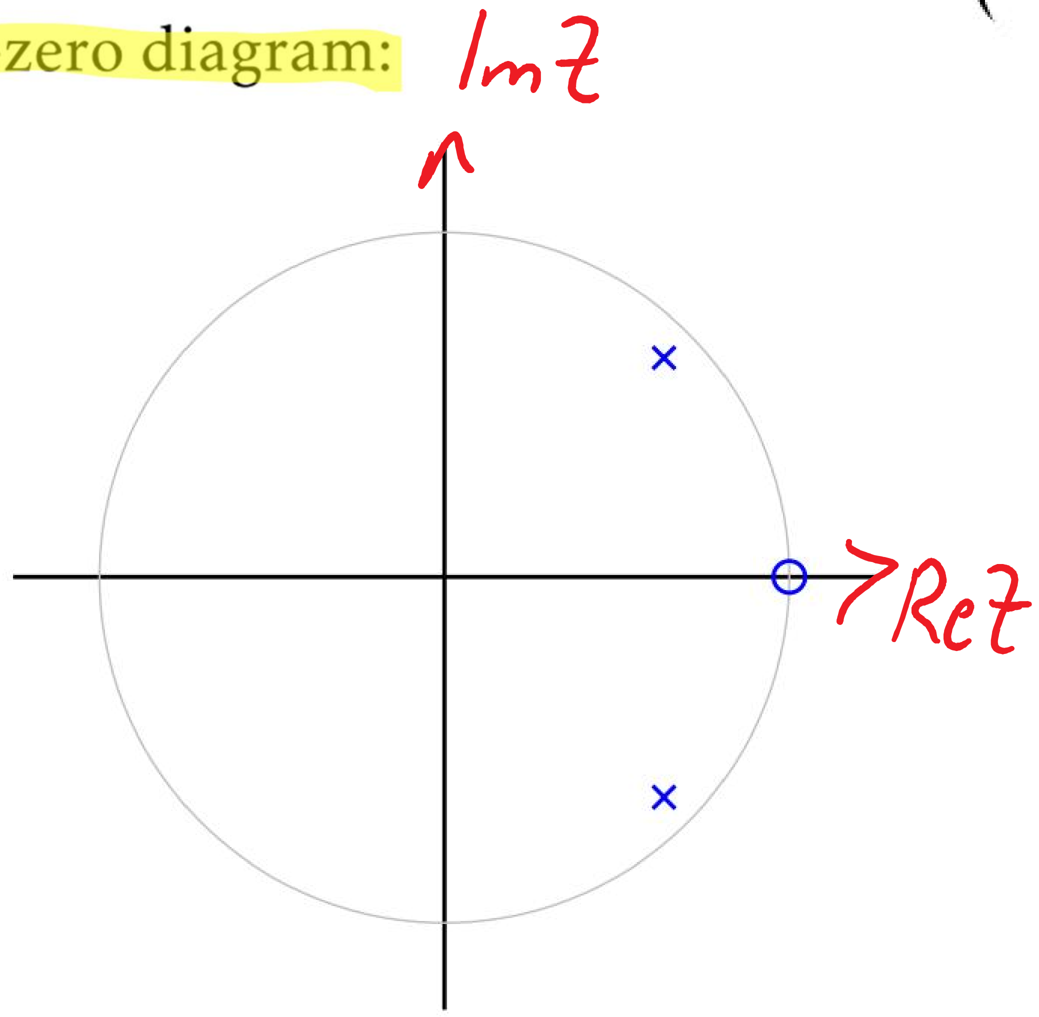
$$H(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}}$$

$$= \frac{z - 1}{z^2 - 1.27z + 0.81}$$

$$= \frac{z - 1}{\left(z - 0.9\mathrm{e}^{\mathrm{j}\frac{\pi}{4}}\right)\left(z - 0.9\mathrm{e}^{-\mathrm{j}\frac{\pi}{4}}\right)}$$

$$H(z) = \frac{z-1}{\left(z - 0.9e^{j\frac{\pi}{4}}\right)\left(z - 0.9e^{-j\frac{\pi}{4}}\right)}$$

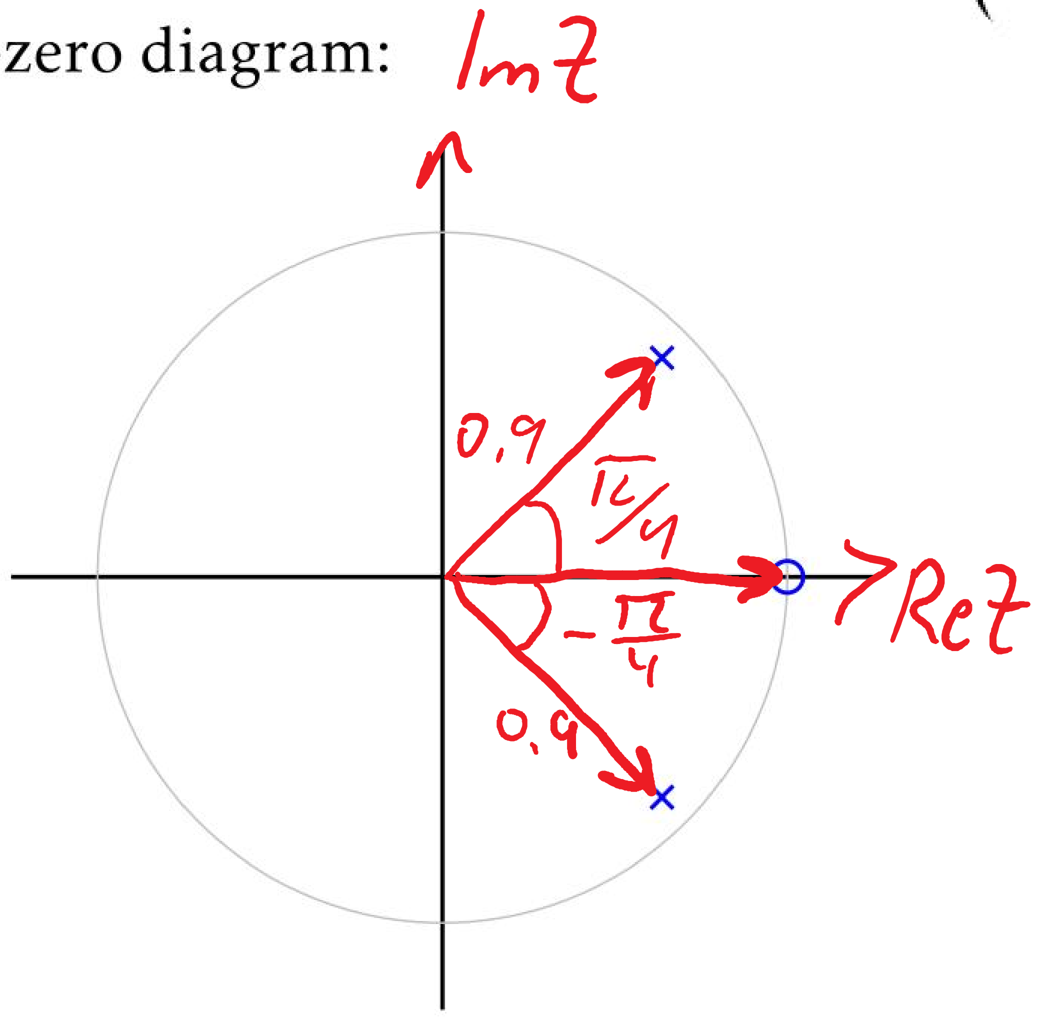
Pole-zero diagram:



Filter response	When...
$H(z) = 0$	z is a zero.
$H(z) = \infty$	z is a pole.
$H(z) \approx 0$	z is close to a zero.
$H(z) \gg 1$	z is close to a pole.

$$H(z) = \frac{z-1}{\left(z-0.9e^{j\frac{\pi}{4}}\right)\left(z-0.9e^{-j\frac{\pi}{4}}\right)}$$

Pole-zero diagram:



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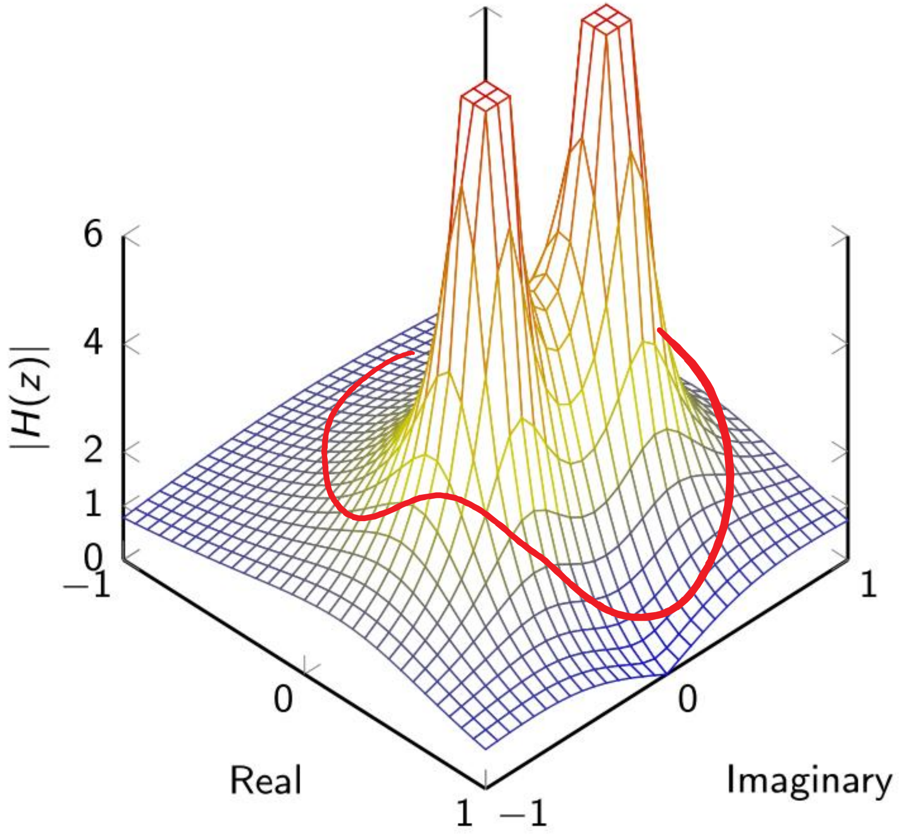
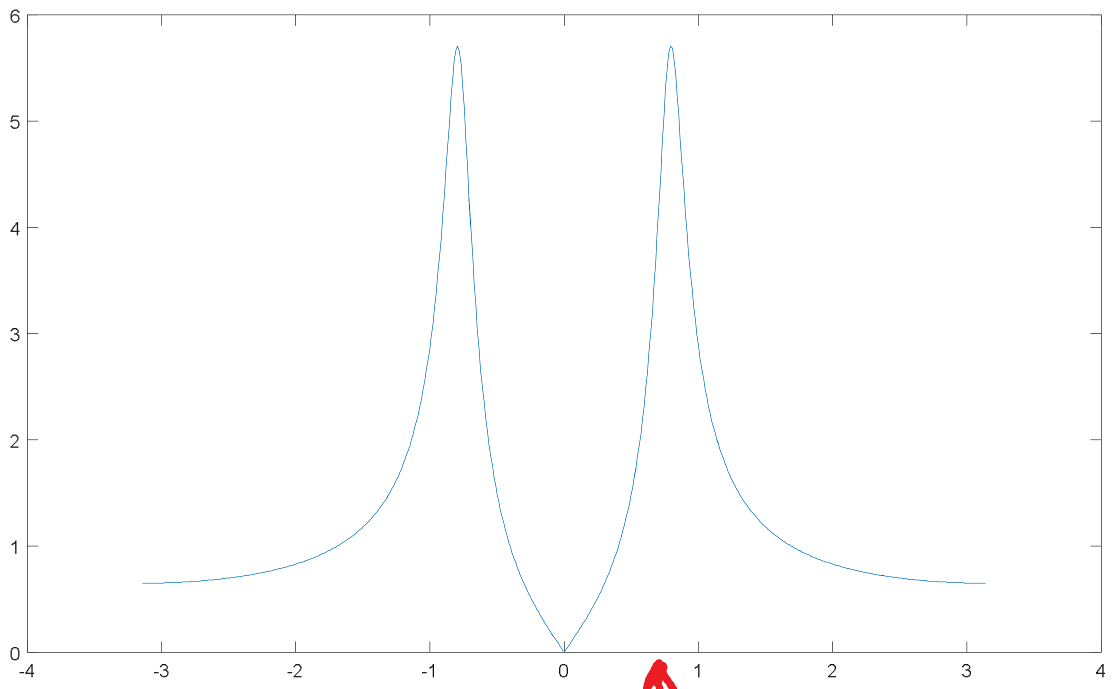
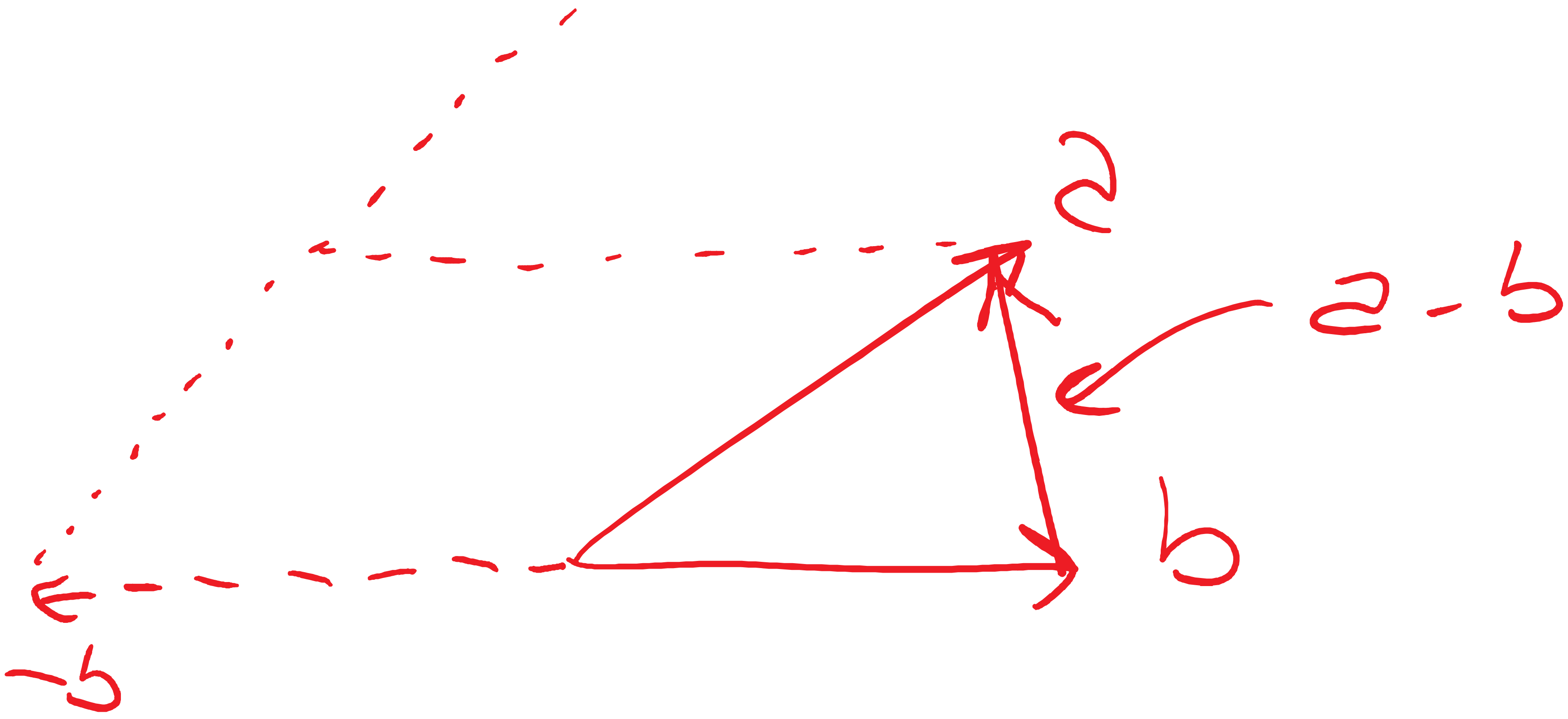
$$H(z) = \frac{z-1}{\left(z - 0.9e^{j\frac{\pi}{4}}\right)\left(z - 0.9e^{-j\frac{\pi}{4}}\right)}$$



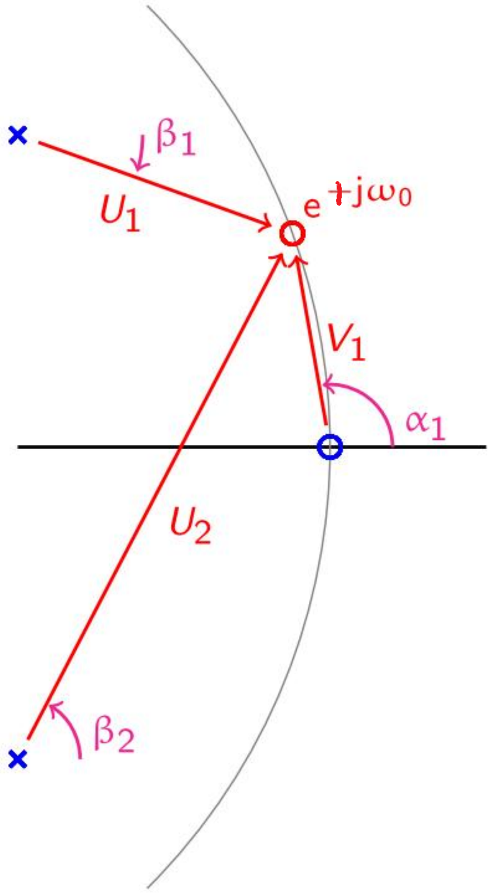


w_0

$H(w_0) = ?$



$$a - b = a + (-b)$$



$$H(z) = \frac{z-1}{\left(z-0.9\mathrm{e}^{\mathrm{j}\frac{\pi}{4}}\right)\left(z-0.9\mathrm{e}^{-\mathrm{j}\frac{\pi}{4}}\right)}$$

$$H(\omega_0) = \frac{\overbrace{e^{+j\omega_0} - 1}^{V_1}}{\underbrace{(e^{+j\omega_0} - \overset{0.9}{e^{j\pi/4}})}_{U_1}\underbrace{(e^{+j\omega_0} - \overset{0.9}{e^{-j\pi/4}})}_{U_2}}$$

The amplitude response is:

$$|H(\omega_0)| = \frac{|V_1|}{|U_1| \cdot |U_2|}$$

The phase response is:

$$\angle H(\omega_0) = \angle V_1 - \angle U_1 - \angle U_2 = \alpha_1 - \beta_1 - \beta_2$$

z-transform of second order system

Sine

$$h(n) = r^n \cdot \sin(\omega n) u(n)$$

$$= r^n \cdot \frac{1}{2\mathrm{j}} \cdot \left(\mathrm{e}^{\mathrm{j}\omega n} - \mathrm{e}^{-\mathrm{j}\omega n}\right) u(n)$$

$$H(z) = \frac{1}{2\mathrm{j}} \cdot \left(\frac{1}{1 - r\mathrm{e}^{\mathrm{j}\omega} z^{-1}} - \frac{1}{1 - r\mathrm{e}^{-\mathrm{j}\omega} z^{-1}}\right)$$

$$= \frac{r \sin(\omega) z^{-1}}{1 - 2r\cos(\omega) z^{-1} + r^2 z^{-2}}$$

Cosine

$$h(n) = r^n \cdot \cos(\omega n) u(n)$$

$$= r^n \cdot \frac{1}{2} \cdot \left(\mathrm{e}^{\mathrm{j}\omega n} + \mathrm{e}^{-\mathrm{j}\omega n}\right) u(n)$$

$$H(z) = \frac{1}{2} \cdot \left(\frac{1}{1 - r\mathrm{e}^{\mathrm{j}\omega} z^{-1}} + \frac{1}{1 - r\mathrm{e}^{-\mathrm{j}\omega} z^{-1}}\right)$$

$$= \frac{1 - r\cos(\omega) z^{-1}}{1 - 2r\cos(\omega) z^{-1} + r^2 z^{-2}}$$

Example

We had

$$H(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}}$$

$$= z^{-1} \cdot \frac{1 - z^{-1} - r \cdot \cos(\omega_0) z^{-1} + r \cdot \cos(\omega_0) z^{-1}}{1 - 1.27z^{-1} + 0.81z^{-2}}$$

$$H(z) = z^{-1} \cdot \frac{1 - r\cos(\omega_0)z^{-1} + \left(r\cos(\omega_0)z^{-1} - z^{-1}\right) \cdot \underbrace{\frac{r\sin(\omega_0)}{r\sin(\omega_0)}}_{=1}}{1 - 1.27z^{-1} + 0.81z^{-2}}$$

$$H(z) = z^{-1} \cdot \frac{1 - r\cos(\omega_0)z^{-1} + \left(r\cos(\omega_0)z^{-1} - z^{-1}\right) \cdot \frac{r\sin(\omega_0)}{r\sin(\omega_0)}}{1 - 1.27z^{-1} + 0.81z^{-2}}$$

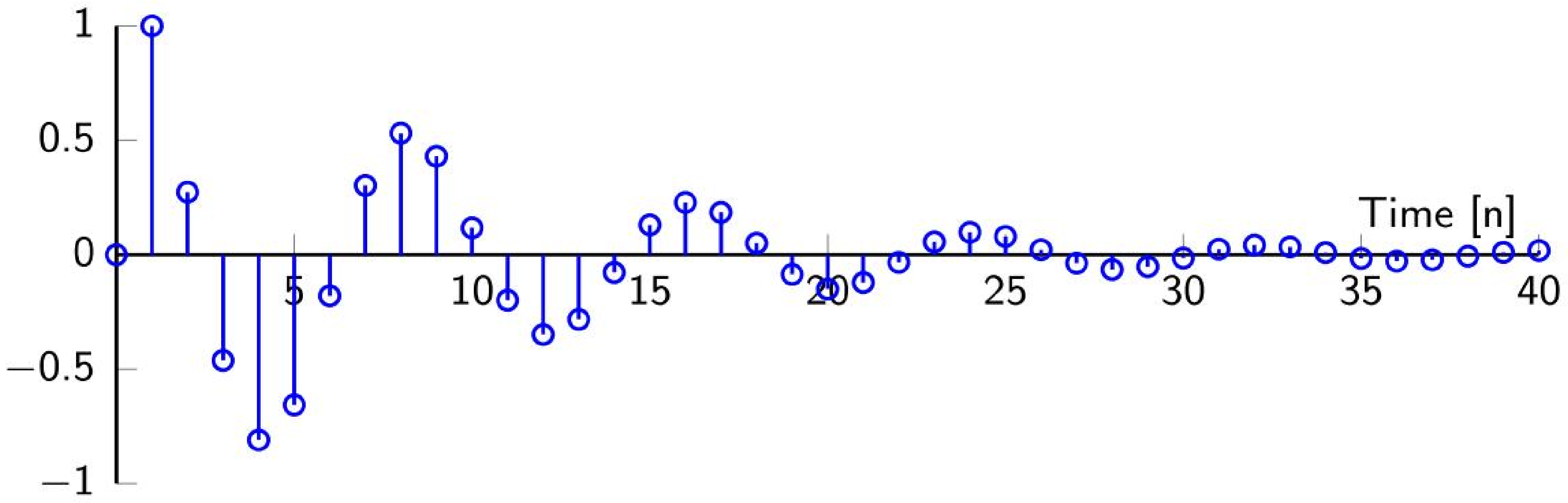
$$r^{n-1} \cdot \left[\cos(\omega_0 \cdot (n-1)) \quad + \quad \frac{r\cos(\omega_0) - 1}{r\sin(\omega_0)} \cdot \sin(\omega_0 \cdot (n-1))\right] u(n-1)$$

(Identify $1.27 = 2r\cos(\omega_0)$ and $0.81 = r^2$. Therefore $\omega_0 = \pi/4$ and $r = 0.9$.)

To summarize, we have the inverse z-transform given by,

$$h(n) = r^{n-1} \cdot \left[\cos(\omega_0 \cdot (n-1)) + \frac{r\cos(\omega_0) - 1}{r\sin(\omega_0)} \cdot \sin(\omega_0 \cdot (n-1)) \right] \cdot u(\overset{n-1}{\cancel{n}})$$

$$= 0.9^{n-1} \cdot \left[\cos\left(\frac{\pi}{4} \cdot (n-1)\right) - 0.57 \sin\left(\frac{\pi}{4} \cdot (n-1)\right) \right] \cdot u(\overset{n-1}{\cancel{n}})$$



Stability

BIBO-stability (bounded input, bounded output)

A sufficient requirement is that

$$\boxed{\sum_n |h(n)| < \infty}$$

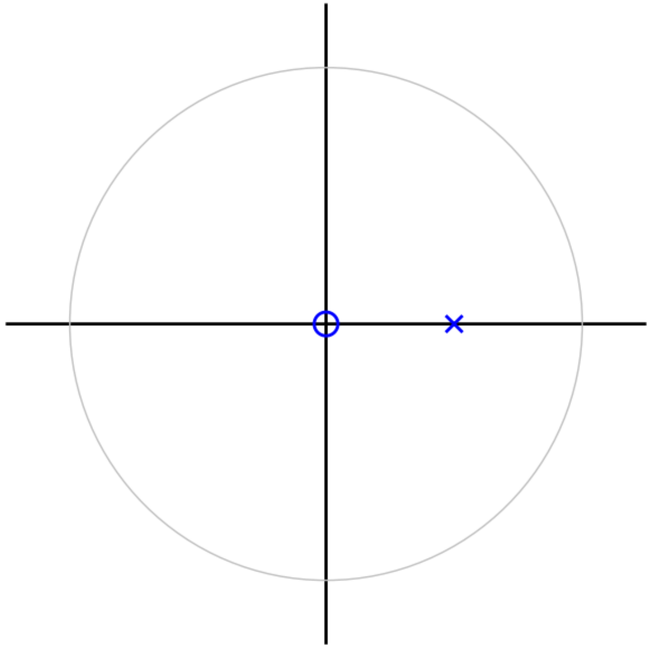
FIR-filter

FIR filters have all their poles at the origin and are always BIBO-stable: $h(n)$ has a finite duration and $\sum_n |h(n)|$ is always limited.

First order IIR-filter

$$H(z) = \frac{1}{1 - az^{-1}} \qquad \Leftrightarrow \qquad h(n) = a^n u(n)$$

Pole in $p_1 = a$.



Stable if $|a| < 1$, or equivalently if the pole lies inside the unit circle.

Second order IIR-filter

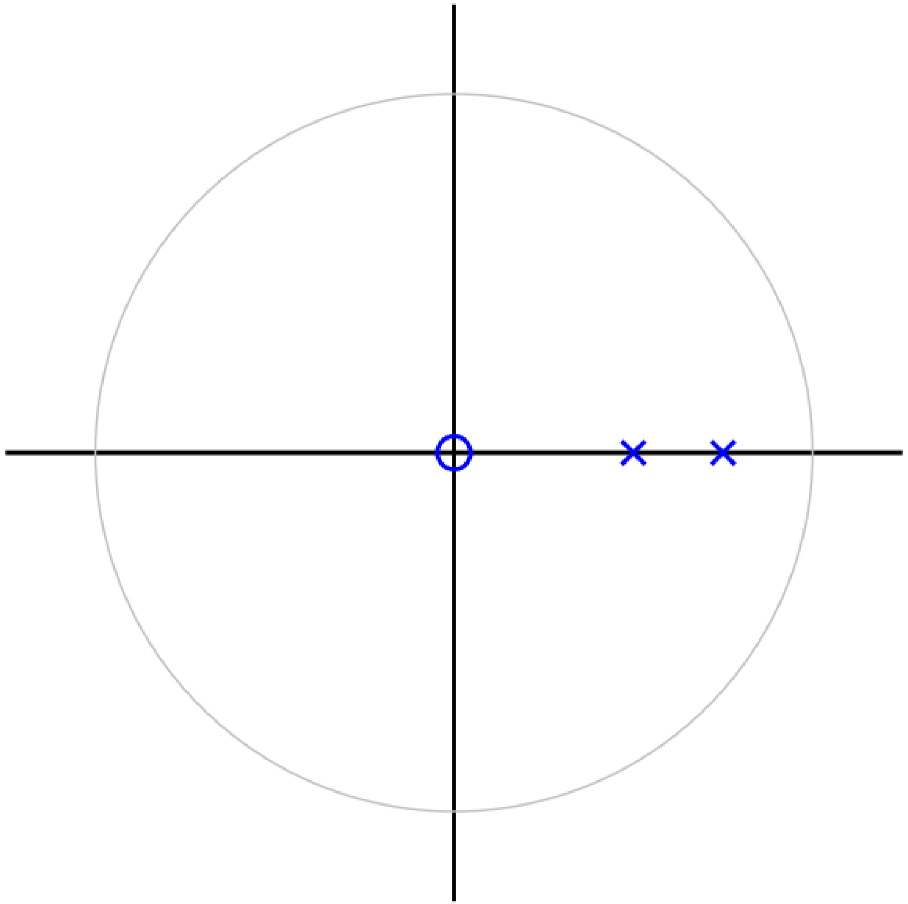
Real poles:

$$H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})}$$

$$= \frac{A}{1 - p_1 z^{-1}} + \frac{B}{1 - p_2 z^{-1}}$$

$$h(n) = (Ap_1^n + Bp_2^n) \cdot u(n)$$

Stable if $|p_1| < 1$ and $|p_2| < 1$.



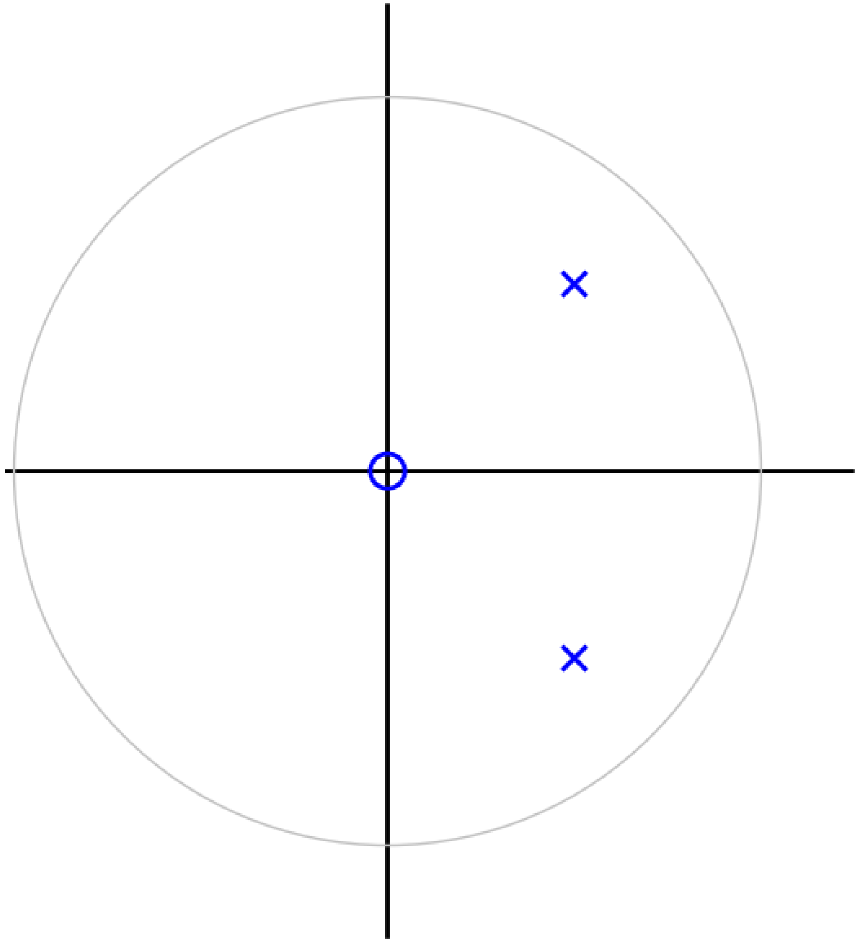
Second order IIR-filter

Complex conjugated poles: Two poles where $p_{1,2} = re^{\pm j\omega_0}$:

$$H(z) = \frac{1}{(1-p_1 z^{-1})(1-p_2 z^{-1})}$$

$$= \frac{1}{1-(p_1+p_2)z^{-1}+p_1 p_2 z^{-2}}$$

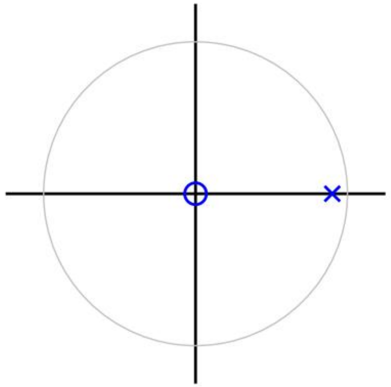
$$= \frac{1}{1-2r\cos(\omega_0)z^{-1}+r^2 z^{-2}}$$

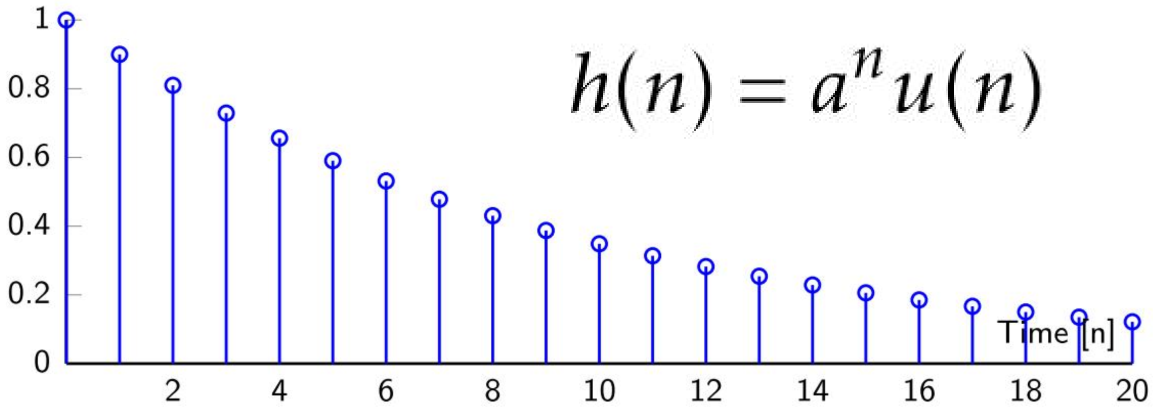


$$h(n) = [Ar^n \sin(\omega_0 n) + Br^n cos(\omega_0 n)]u(n)$$

Stable if $|p_1| < 1$ and $|p_2| < 1$, or equivalently if $|r| < 1$.

Examples of relation between pole/zero-plots and impulse response

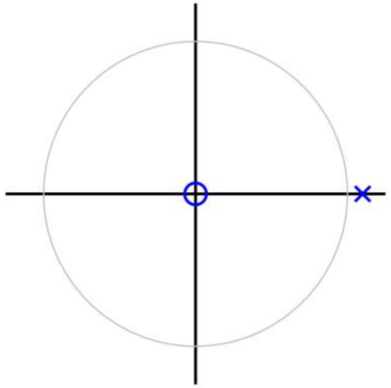


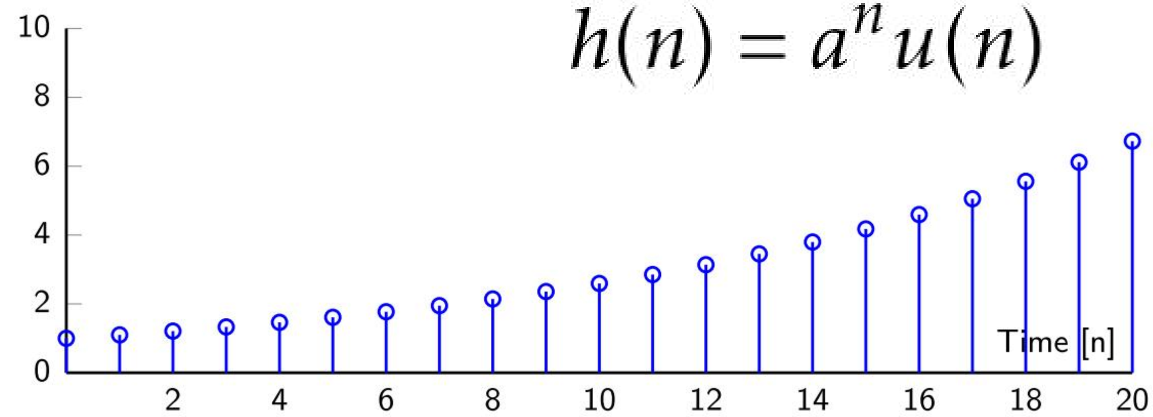


$$h(n) = a^n u(n)$$

$$|a| < 1$$

Exponentially increasing step



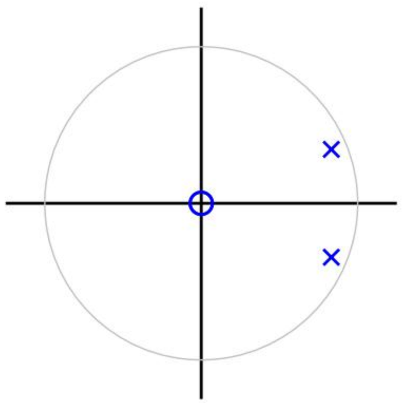


$$h(n) = a^n u(n)$$

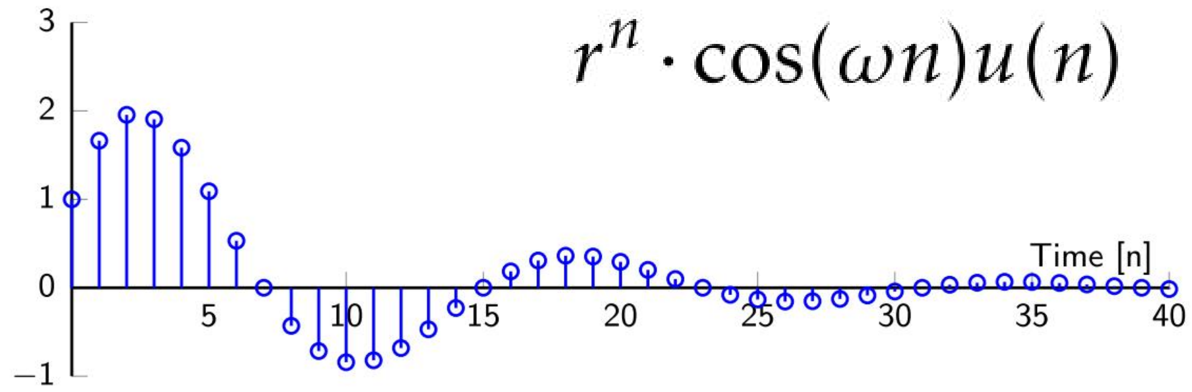
$$|a| > 1$$

Examples of relation between pole/zero-plots and impulse response

Damped oscillator

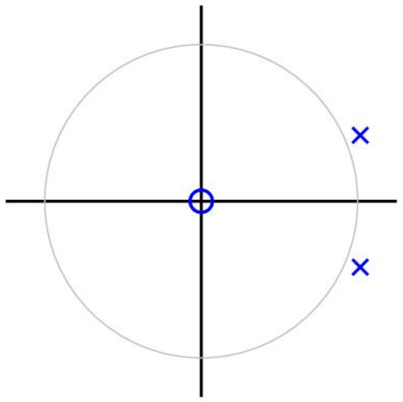


$$r^n \cdot \cos(\omega n) u(n)$$

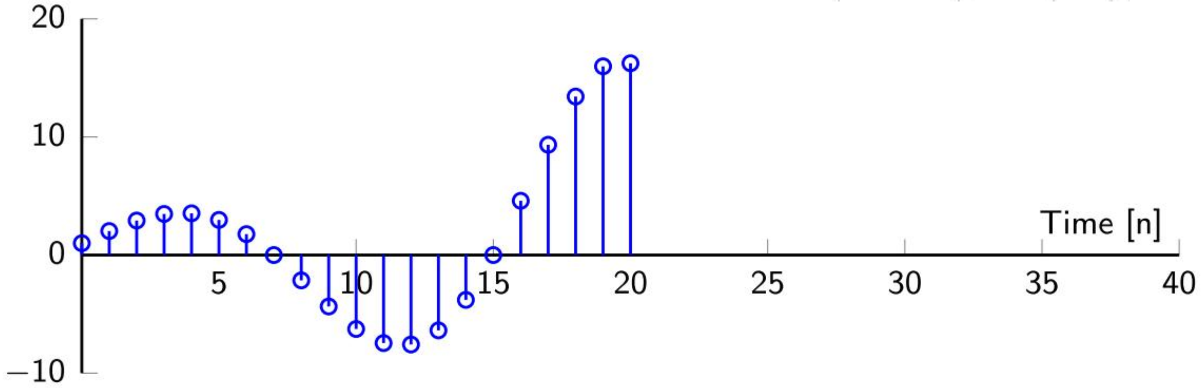


$r < 1$

Unstable oscillator



$$r^n \cdot \cos(\omega n) u(n)$$



$r > 1$

Appendix: (for those of you that have background knowledge in Laplace-transforms)

Analog $H(s)$ and discrete $H(z)$

Analog: Stable if all poles are in the left half-plane.

$$H(s) = \int_t h(t)e^{-st}\,dt$$

Discrete: Stable if all poles lie within the unit circle.

$$H(z) = \sum_n h(n)z^{-n}$$

where $z \sim e^s$.

Example

$$p_{\text{analog}} = -0.5 \pm 0.5j$$

$$\begin{aligned} p_{\text{digital}} &\sim e^{-0.5 \pm j0.5} \\ &= e^{-0.5}e^{\pm j0.5} \\ &= 0.6e^{\pm j0.16\pi} \end{aligned}$$

$$s = \sigma + j\Omega \qquad \Leftrightarrow \qquad z = r \cdot e^{j\omega}$$

Solving general differential equations

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \tag{52}$$

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) \tag{53}$$

$$Y(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \cdot X(z) \tag{54}$$

$$= b_0 \cdot \frac{z^{-M}}{z^{-N}} \cdot \frac{(z - z_1)\cdots(z - z_M)}{(z - p_1)\cdots(z - p_N)} \cdot X(z) = H(z)X(z) \tag{55}$$

where z_i are the zeros (roots to the numerator polynomial) and p_i are the poles (roots to the ~~numerator~~ polynomial). Inverse transform $Y(z)$ by partial fraction expansion and the solution is given by the resulting $y(n)$.

Denominator