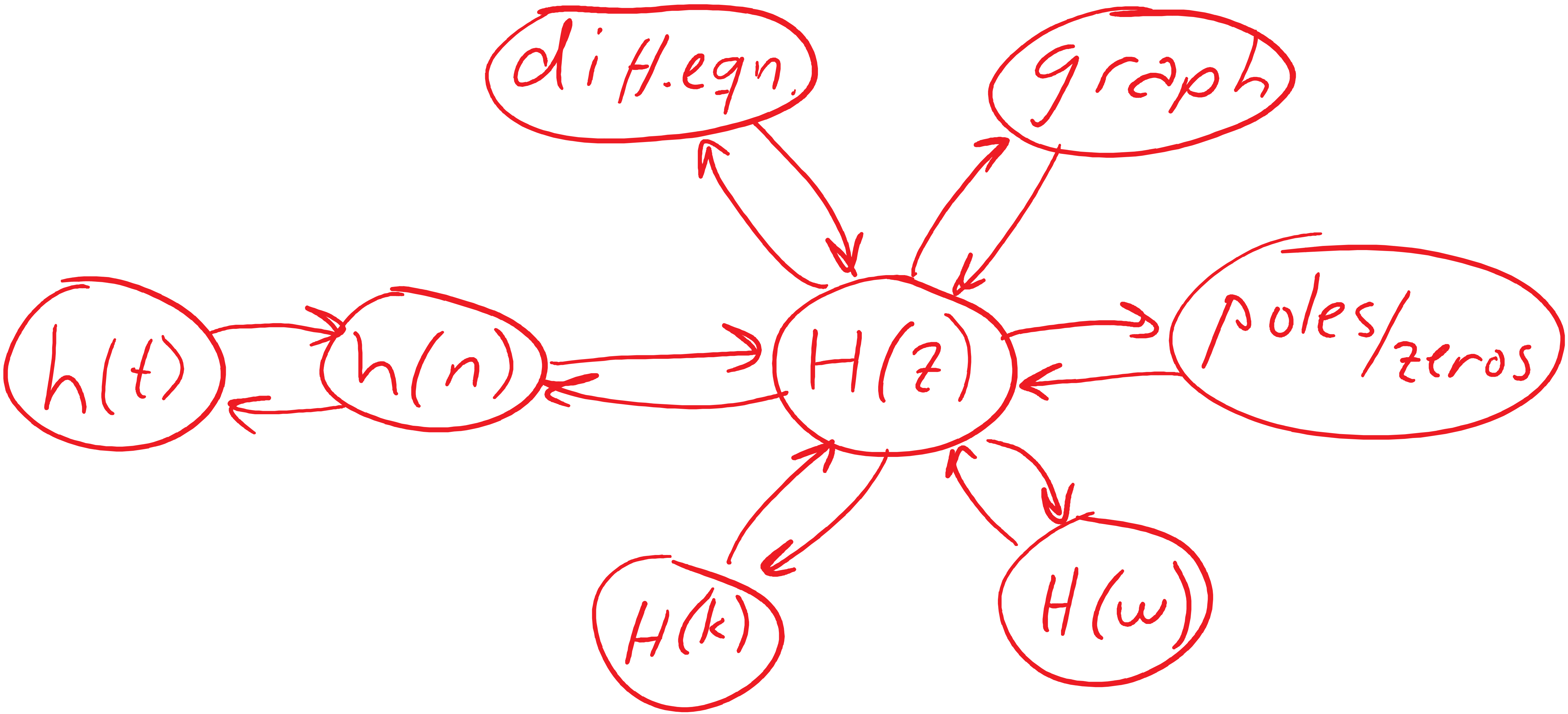
Lecture 5

Digital Signal Processing

Chapter 4

Fourier transforms of analog and digital signals


diff.eqn.
graph
h(t)
h(n)
H(z)
poles/zeros
H(k)
H(ω)

Transforms in Digital Signal Processing

Analog signals

- Fourier transform of analog signals, FT (Laplace transform)

Discrete signals (sampled, digital signals)

- Fourier transform of digital signals, DTFT, chapter 4
- Discrete Fourier transform (DFT, FFT), chapter 7
- z-transform of digital signals, chapter 3.

Fourier series expansion

- Fourier series expansion of periodic signals

Fourier transform (page 236-238)

Fourier transform of analog signals

Trans

$$X(F) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{j}2\pi F t}\,\mathrm{d}t$$

Convergence if $x(t)$ is stable:

$$\int |x(t)|\,\mathrm{d}t < \infty$$

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{j}\Omega t}\,\mathrm{d}t \qquad [\text{where } \Omega = 2\pi F]$$

Inv Trans

$$x(t) = \int_{-\infty}^{\infty} X(F)\mathrm{e}^{\mathrm{j}2\pi F t}\,\mathrm{d}F = \frac{1}{2\pi}\cdot\int_{-\infty}^{\infty} X(\Omega)\mathrm{e}^{\mathrm{j}\Omega t}\,\mathrm{d}\Omega$$

Fourier transform of discrete signals, DTFT (page 251)

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n}$$

Convergence if $x(n)$ is stable:

$$\sum_n |x(n)| < \infty$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad [\text{where } \omega = 2\pi f]$$

$$x(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f) e^{j2\pi f n} \, df = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \, d\omega$$

Compare the z-transform and the Fourier transform.

$$X(z) = \sum_n x(n) z^{-n}$$

$$X(\omega) = \sum_n x(n) e^{-j\omega n}$$

Important:

If $x(n)$ is causal and stable

$$\boxed{X(\omega) = X(z \mid z = e^{j\omega})}$$

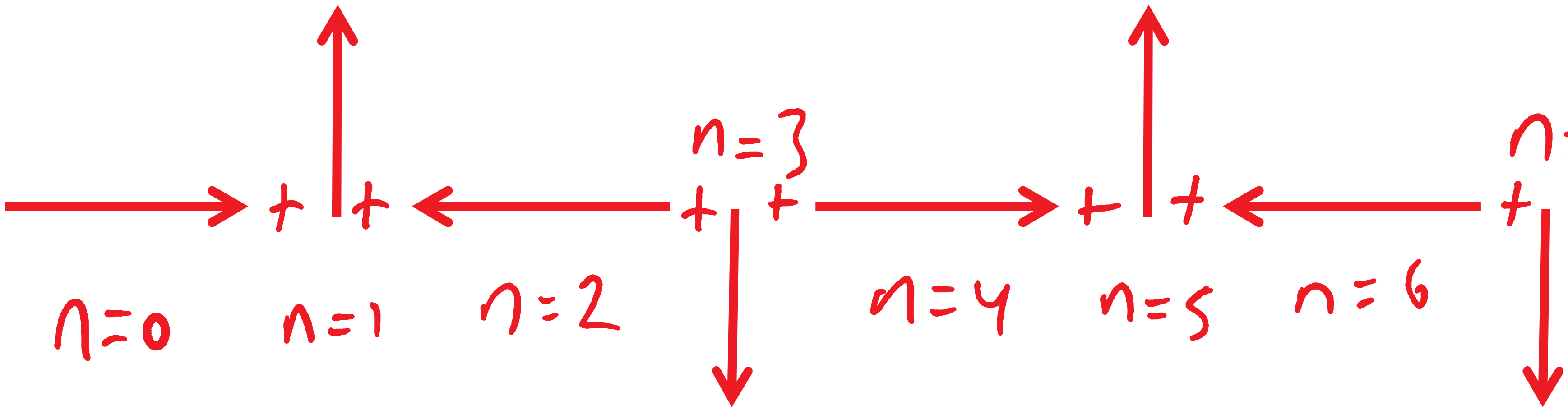
EX: Illustration of the Discrete-Time Fourier transform
(A similar illustration can be done for the continuous case)

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)\underbrace{e^{-j2\pi f n}}_{\text{a rotating vector}}$$

let us look at an exampel where x(n) is also a rotating vector, i.e. $X(n) = e^{j2\pi f_0 n}$

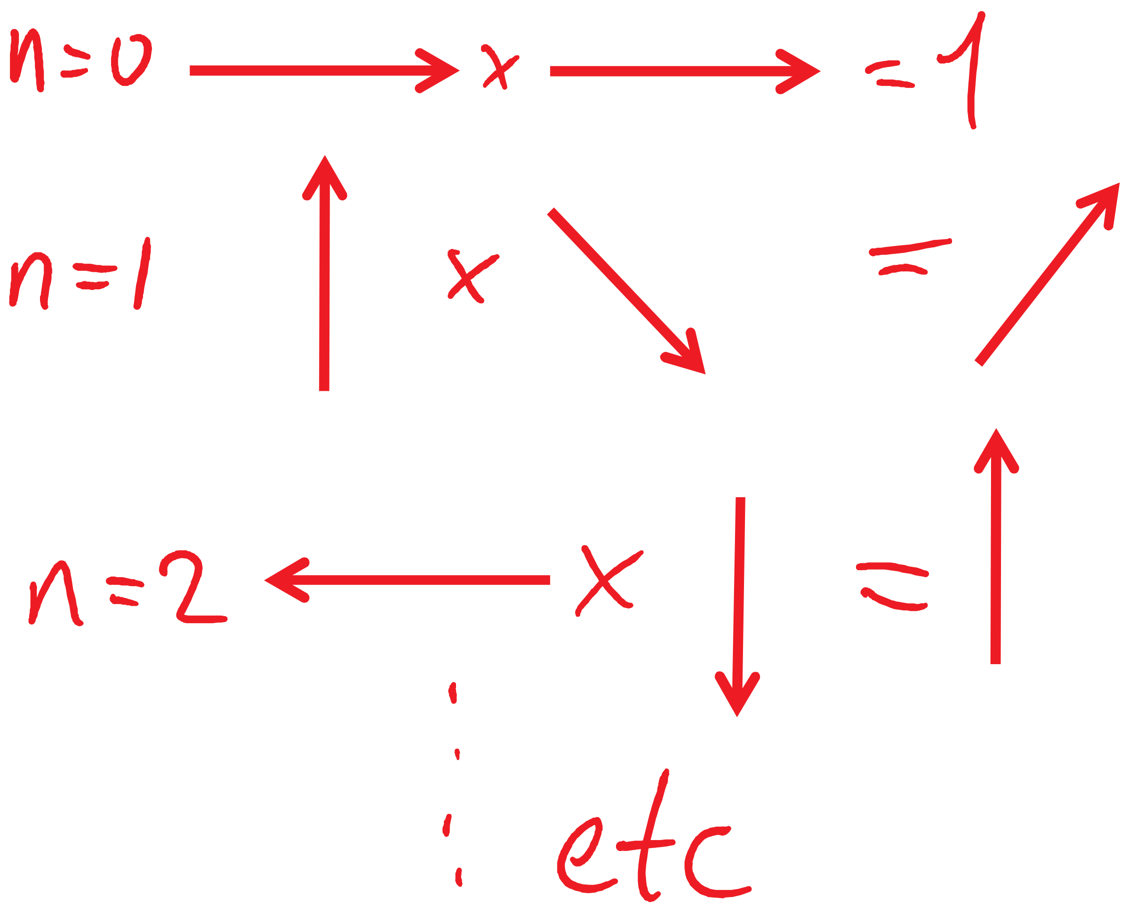
Ex: when the length of the signal is 8, i.e. calculate X(0), X(1/8), X(1/4) and X(1/2), when $f_0 = \frac{1}{4}$

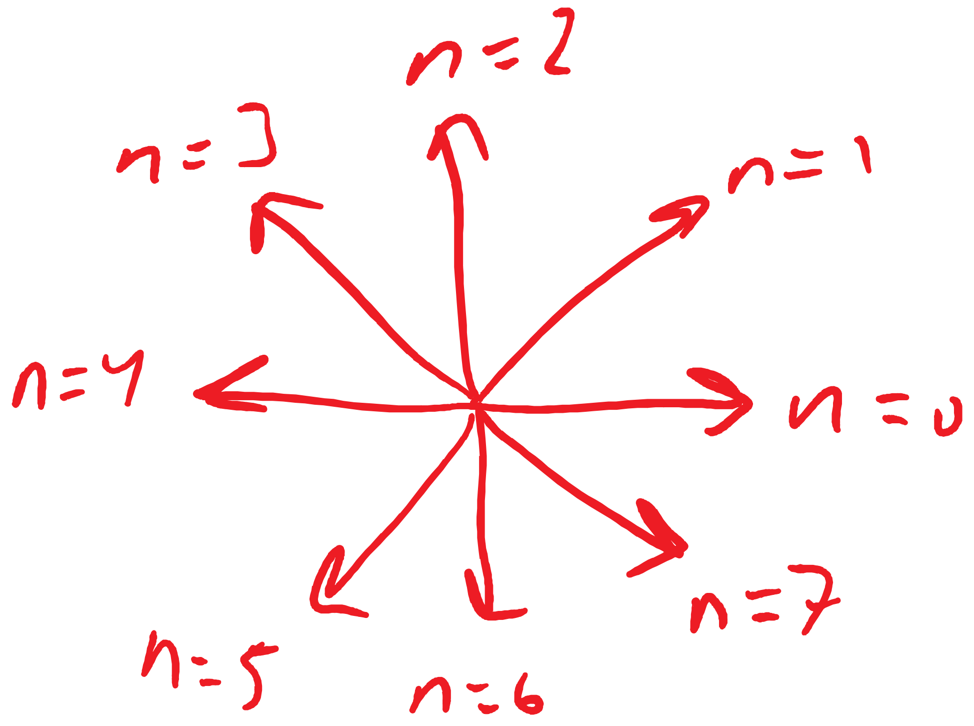
$$X(0) = \sum_{n=0}^{7} e^{j2\pi f_0 n} \underbrace{e^{-j2\pi 0 n}}_{=1}$$



$= 0$

$$X\left(\frac{1}{8}\right) = \sum_{n=0}^{7} e^{j2\pi f_0 n} e^{-j2\pi \frac{1}{8} n} \quad , f_0 = \frac{1}{4}$$





$\Sigma = 0$

$$X\left(\frac{1}{4}\right) = \sum_{n=0}^{7} e^{j2\pi f_0 n} e^{-j2\pi \frac{1}{4} n} \quad , f_0 = \frac{1}{4}$$

$n=0 \longrightarrow \times \longrightarrow =1$

$n=1 \quad \times \quad = \longrightarrow =1$

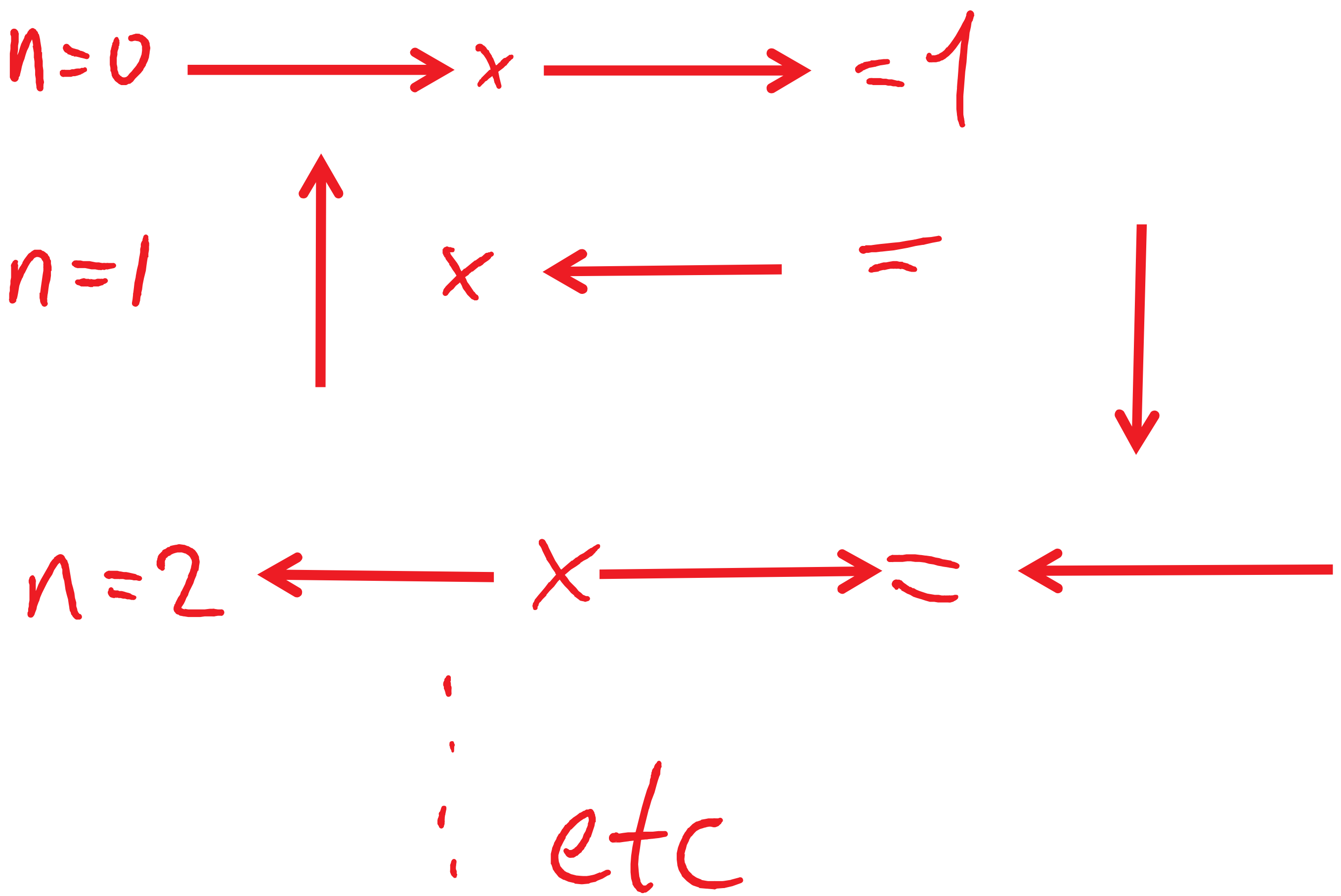
$n=2 \longleftarrow \times \longleftarrow = \longrightarrow =1$

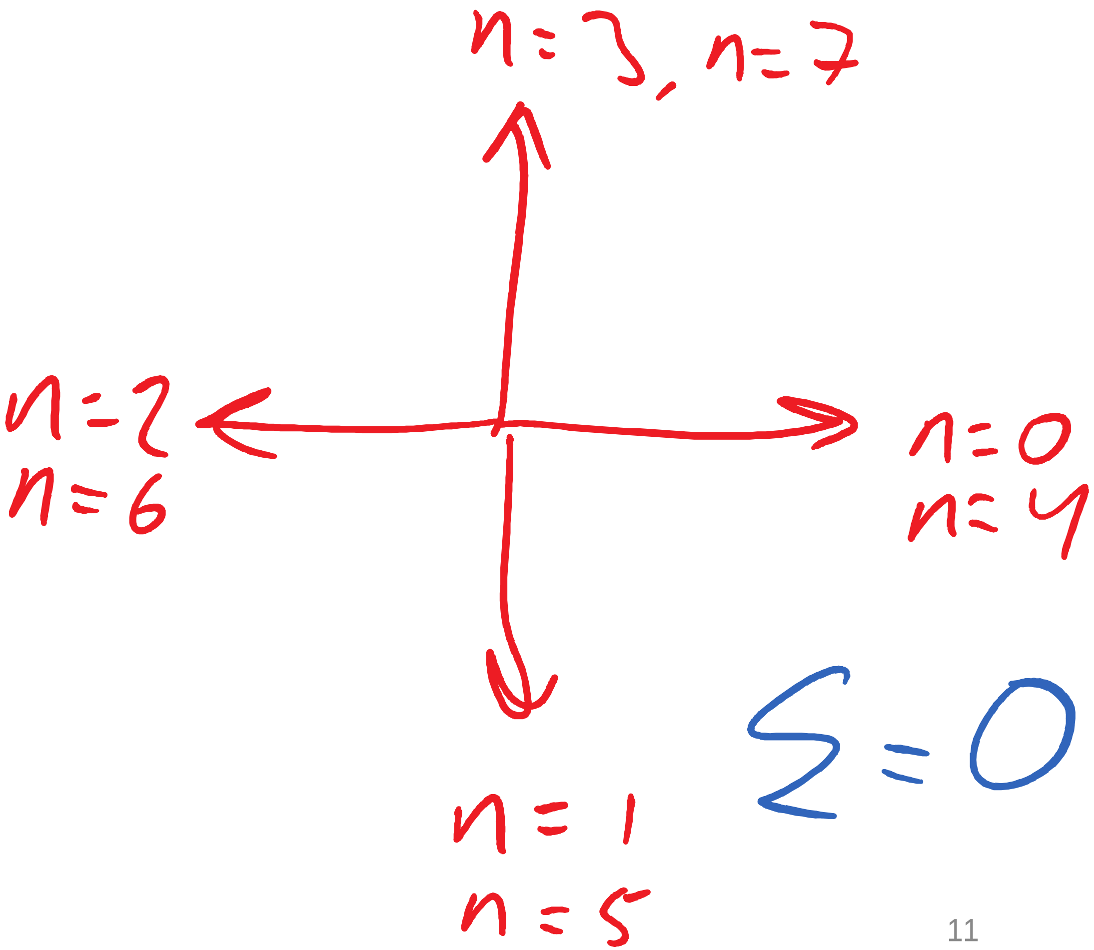
⋮ etc

$1+1+1+1+1+1+1+1$

$= 8$

$$X\left(\frac{1}{2}\right) = \sum_{n=0}^{7} e^{j2\pi f_0 n} e^{-j2\pi \frac{1}{2} n} \quad , f_0 = \frac{1}{4}$$





$\Sigma = 0$

So we got the following;

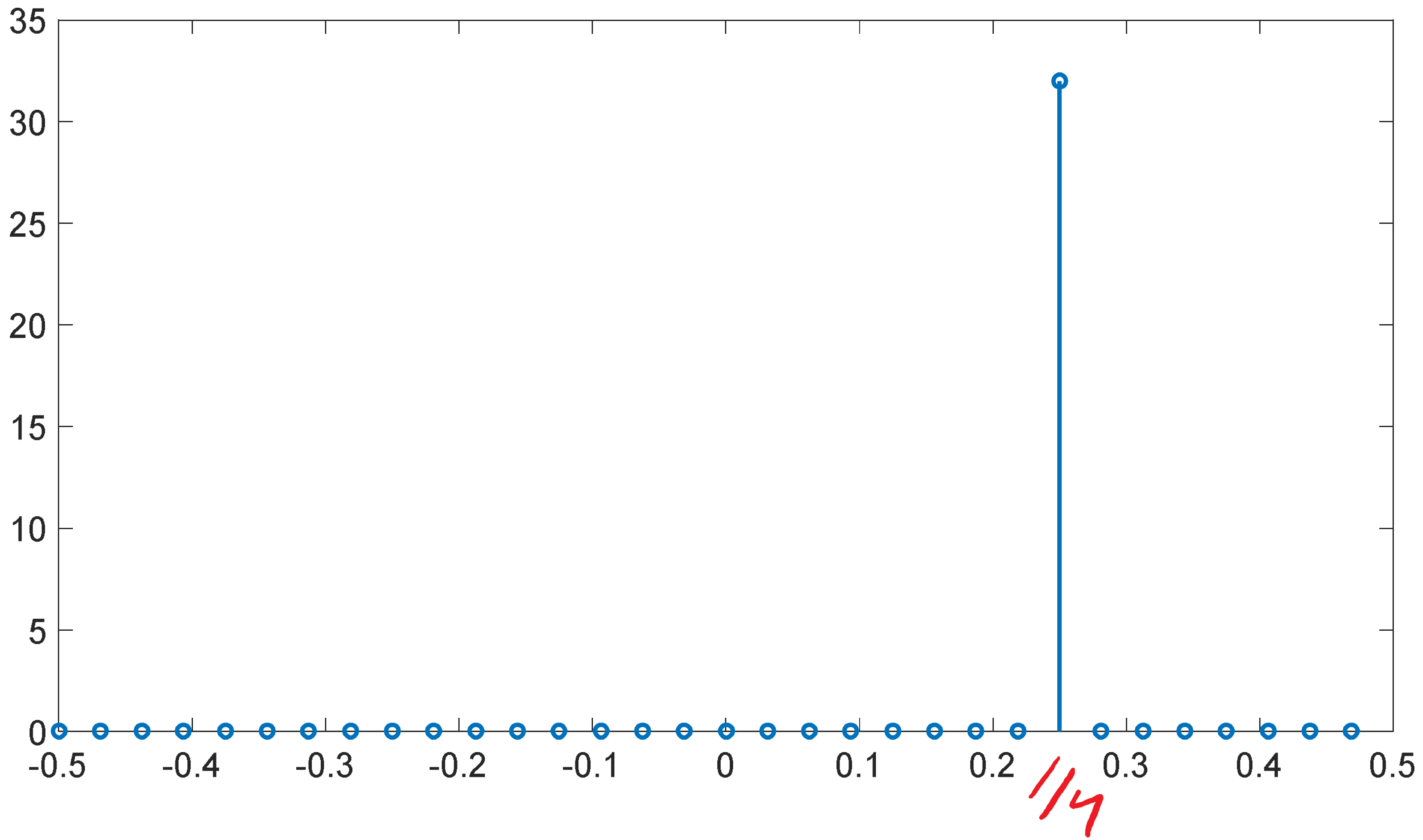
$X(0) = 0$
$X(1/8) = 0$
$X(1/4) = 8$
$X(1/2) = 0$

OBS!

$f_0 = \frac{1}{4}$

Let us take a similar example in Matlab where N=32;

```
N=32; x=exp(i*2*pi*1/4*(0:N-1));
figure,stem(linspace(-0.5,0.5-1/N,N),abs(fftshift(fft(x,N))))
```



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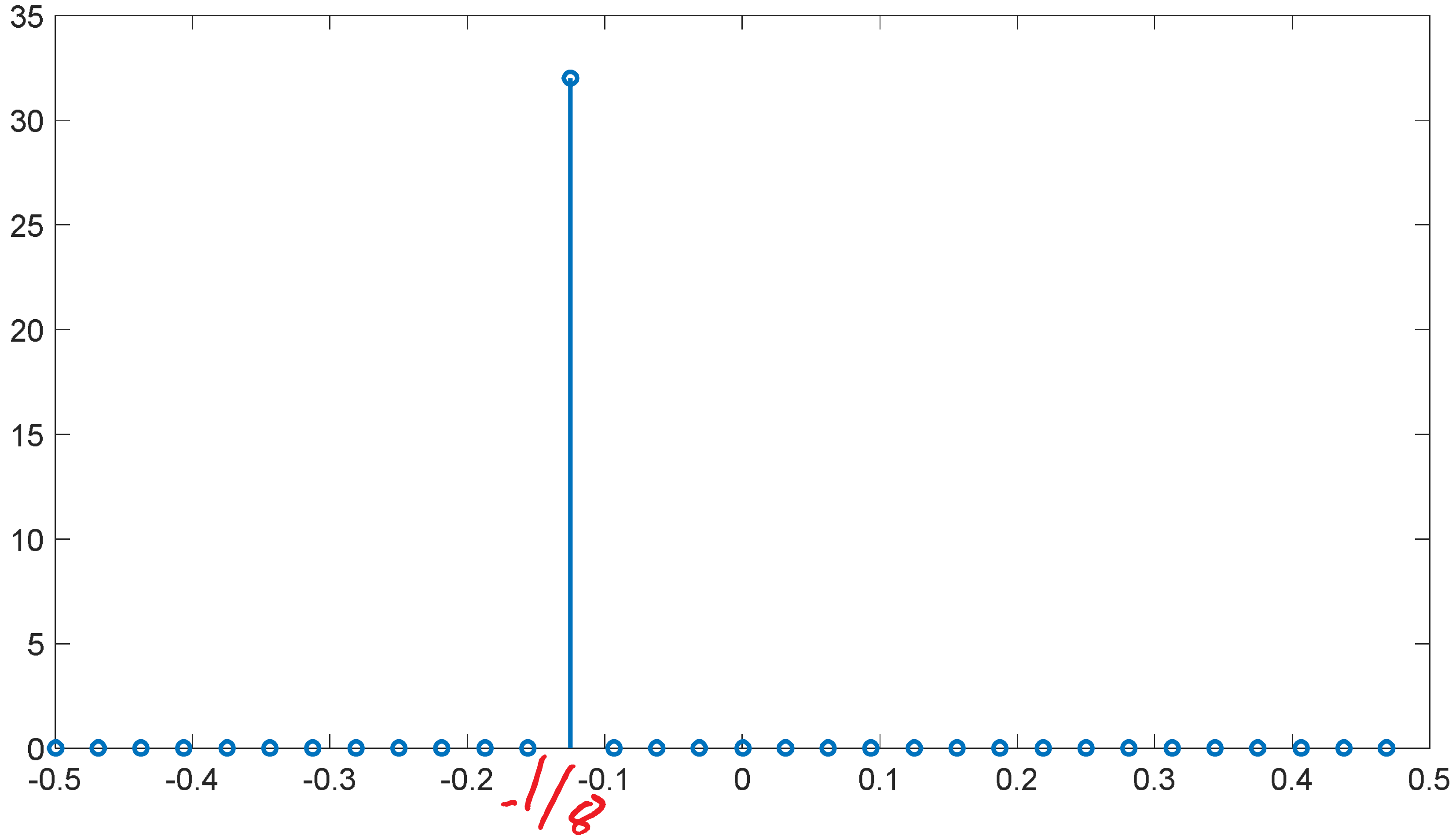
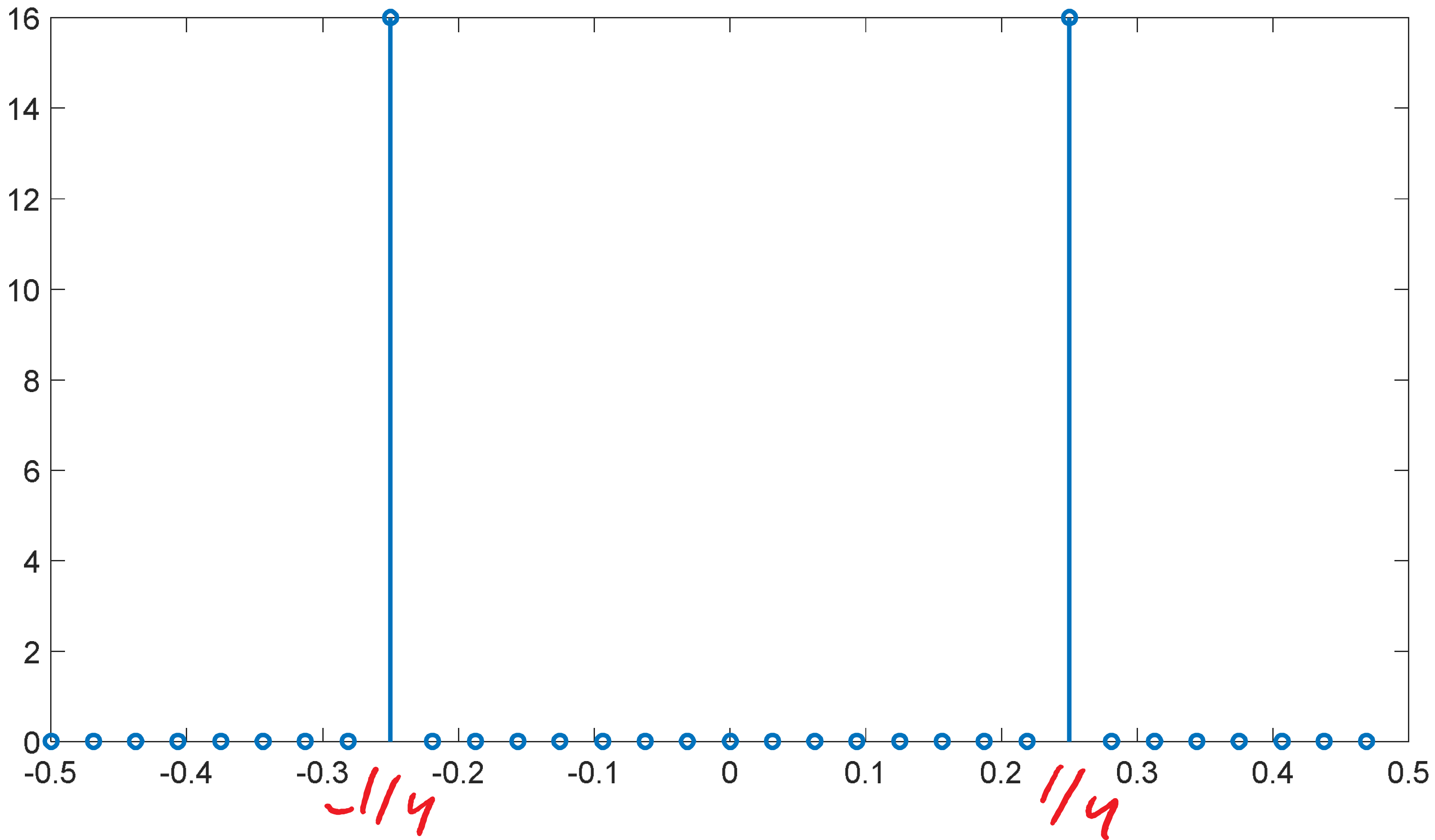
```



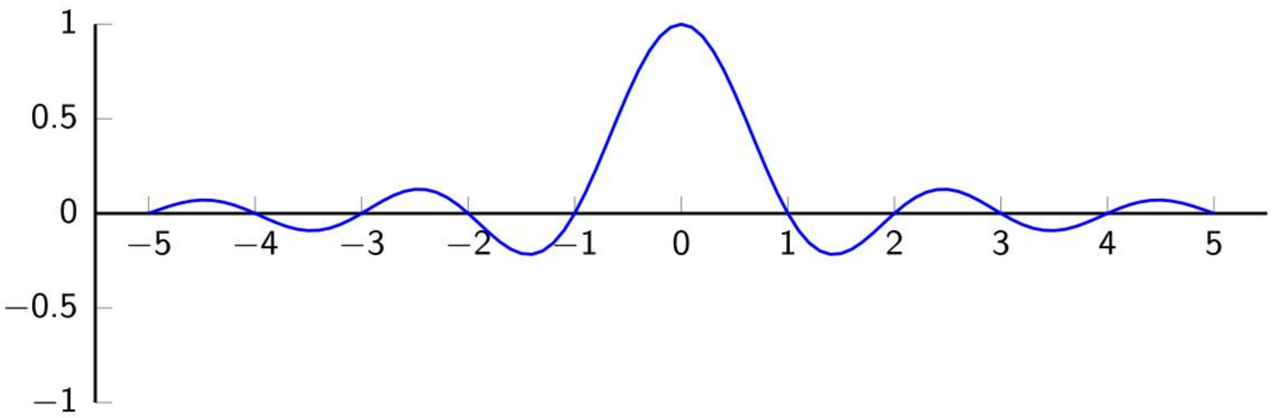
Let us take a similar example in Matlab where N=32;

```
N=32; x=sin(2*pi*1/4*(0:N-1));
figure,stem(linspace(-0.5,0.5-1/N,N),abs(fftshift(fft(x,N))))
```



sinc function

$$\mathrm{sinc}(x) \overset{def}{=} \frac{\sin(\pi x)}{\pi x}$$



where

$$\lim_{x \to 0} \frac{\sin(\pi x)}{\pi x} = 1$$

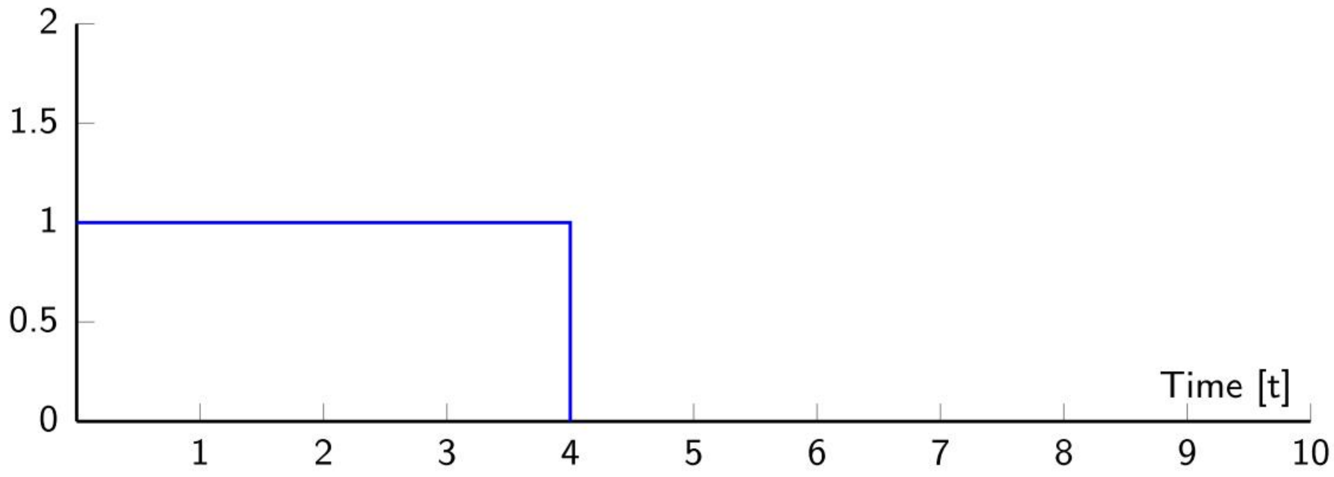
Fourier transform of rectangular pulse (page 257-258)

Analog rectangular pulse (rectangular window)

$$x(t) = \begin{cases} 1 & 0 \leq t < T \\ 0 & \text{otherwise} \end{cases}$$

What is the frequency content of this signal?

The signal $x(t)$ when $T = 4$.



$$X(F) = \int_{-\infty}^{\infty} x(t) \mathrm{e}^{-\mathrm{j}2\pi F t}\, \mathrm{d}t$$

$$= \int_{0}^{T} 1 \cdot \mathrm{e}^{-\mathrm{j}2\pi F t}\, \mathrm{d}F$$

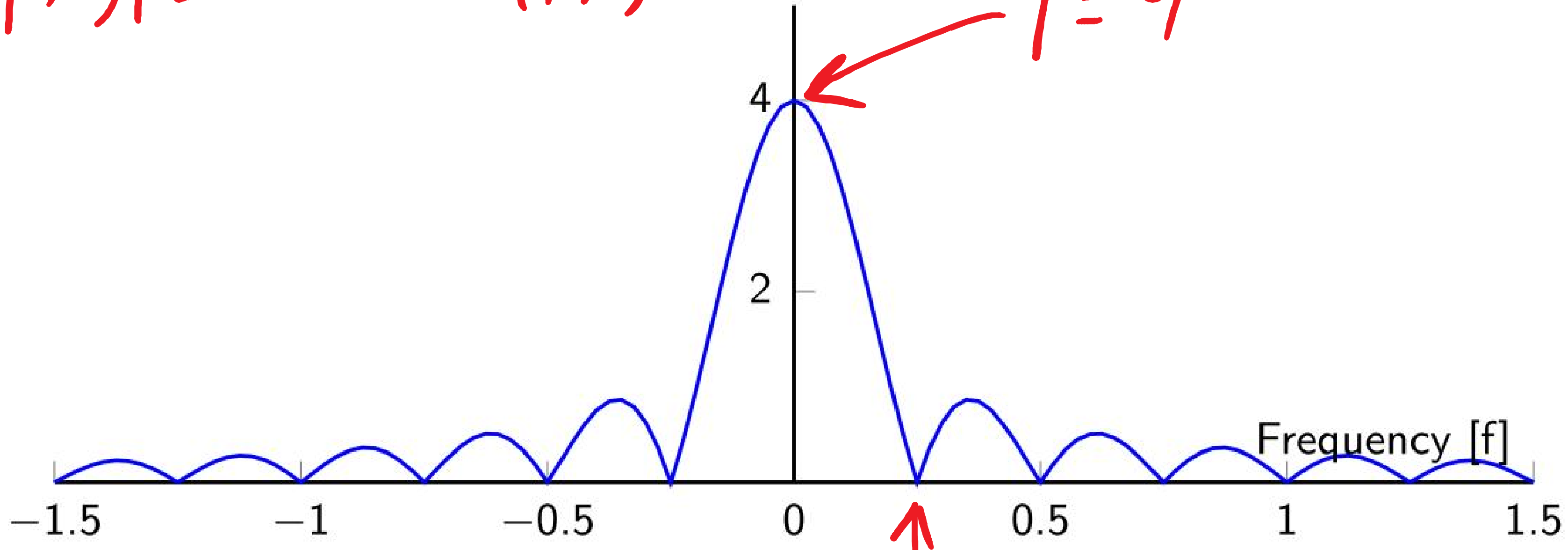
$$= \frac{\mathrm{e}^{-\mathrm{j}2\pi F T} - 1}{-\mathrm{j}2\pi F} = \{\text{simplify}\} =$$

$$= \frac{\mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{T}{2} \cdot F}\left(\mathrm{e}^{\mathrm{j}2\pi \cdot \frac{T}{2} \cdot F} - \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{T}{2} \cdot F}\right)}{\mathrm{j}2\pi F} = T \cdot \underbrace{\frac{\sin\left(2\pi \cdot \frac{T}{2} \cdot F\right)}{2\pi \cdot \frac{T}{2} \cdot F}}_{\mathrm{sinc}(T \cdot F)} \cdot \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{T}{2} \cdot F}$$

The amplitude function $|X(F)|$ $|\cancel{H(F)}|$ of the signal $x(t)$.

$|X(F)| = T \cdot \mathrm{sinc}(TF)$

$T = 4$



[Hz]

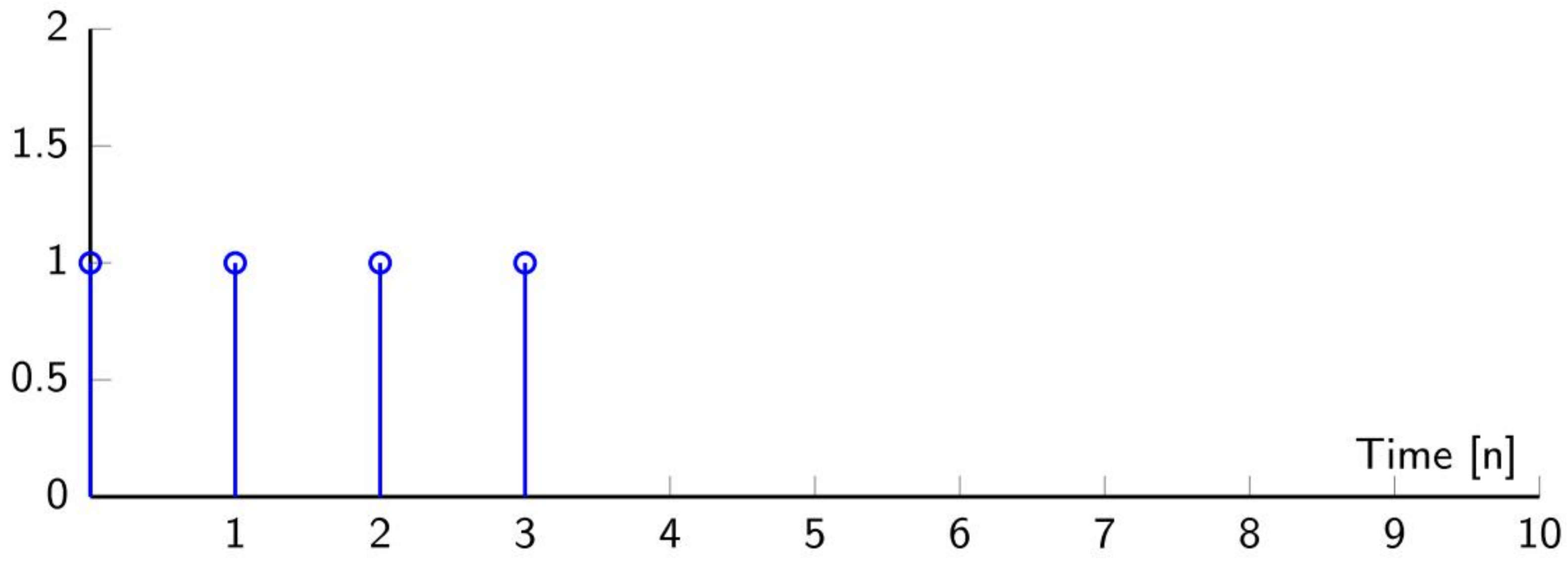
$\frac{1}{T} = \frac{1}{4}$

Discrete rectangular pulse (rectangular window)

$$x(n) = \begin{cases} 1 & 0 \le \overset{n}{\cancel{t}} < \overset{N}{\cancel{T}} \\ 0 & \text{otherwise} \end{cases}$$

What is the frequency content of this signal?

The signal $x(n)$ for $N = 4$.



$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} 1 \cdot e^{-j\omega n} = \{\text{geometric sum}\} =$$

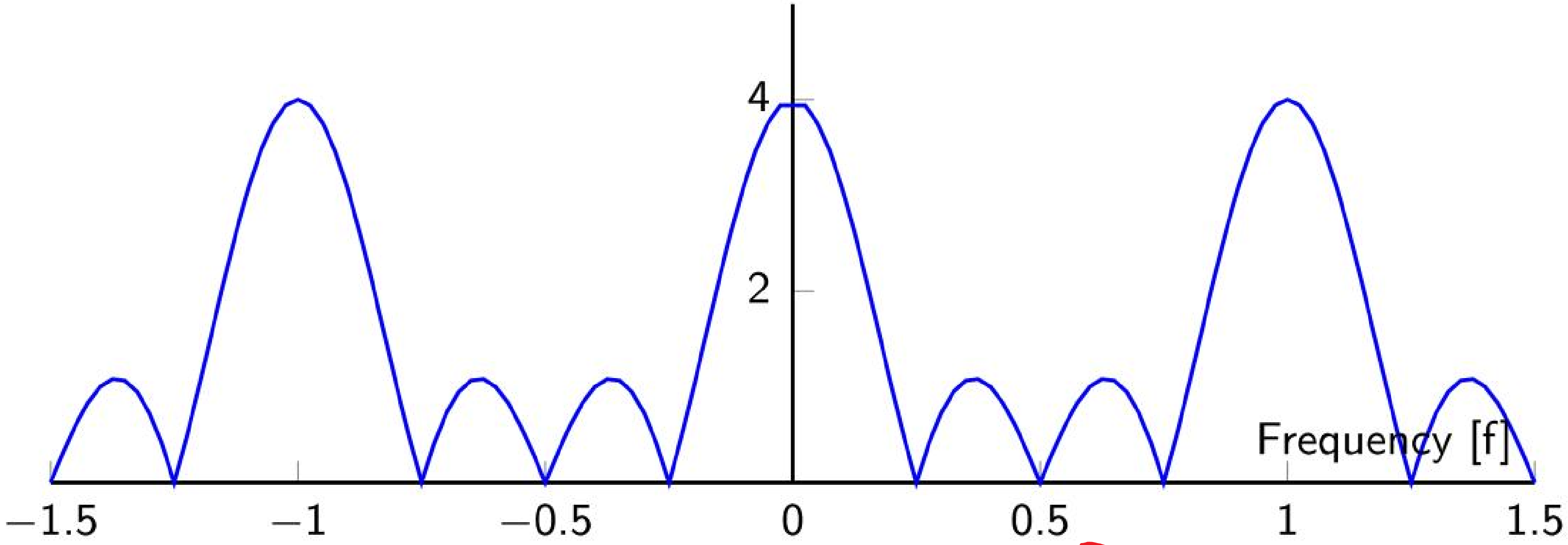
$$= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} = \{\text{simplify}\} =$$

$$= \frac{e^{j\omega N/2}\left(e^{j\omega \cdot \frac{N}{2}} - e^{-j\omega \cdot \frac{N}{2}}\right)}{e^{j\omega/2}\left(e^{j\omega \cdot \frac{1}{2}} - e^{-j\omega \cdot \frac{1}{2}}\right)} = N \cdot \frac{\sin\left(\omega \cdot \frac{N}{2}\right)}{N \sin\left(\omega \cdot \frac{1}{2}\right)} \cdot e^{-j\omega(N-1)/2}$$

$|X(\omega)|$

The amplitude function $|\cancel{H(f)}|$ of the signal $x(n)$.

$$|X(\omega)| = N \cdot \left| \frac{\sin\left(\omega \cdot \frac{N}{2}\right)}{N \sin\left(\omega \cdot \frac{1}{2}\right)} \right|$$



$\omega = -\pi$ $\omega = \pi$

$\boxed{\text{Observe that } X(\omega) \text{ is periodic with the period } f = 1 \text{ or } \omega = 2\pi.}$

Example of transforms (DTFT)

Discrete signal:

$$x(n) = \{ \underset{\substack{\uparrow \\ n=0}}{3} \quad 2 \quad 1 \}$$

$$X(\omega) = 3 + 2e^{-j\omega} + e^{-j2\omega}$$

$$x_1(n) = \{ \underset{\substack{\uparrow \\ n=0}}{0} \quad 3 \quad 2 \quad 1 \} = x(n-1)$$

$$\Rightarrow X_1(\omega) = 0 + 3e^{-j\omega} + 2e^{-j2\omega} + e^{-j3\omega}$$

$$= e^{-j\omega} \cdot \underbrace{\left(3 + 2e^{-j\omega} + e^{-j2\omega}\right)}_{X(\omega)}$$

Delta function:

$$x(n) = \delta(n) = \{ \underset{\substack{\uparrow \\ n=0}}{1} \}$$

$$X(\omega) = 1$$

Time shift:

$$y(n) = x(n - n_0)$$

$$Y(\omega) = e^{-j\omega n_0} \cdot X(\omega)$$

Compare

$$Y(z) = z^{-n_0} \cdot X(z)$$

Convolution becomes multiplication:

$$y(n) = h(n) * x(n) = \sum_k x(k)h(n-k)$$

$$Y(\omega) = \sum_n y(n)e^{-j\omega n}$$

$$= \sum_n \overbrace{\sum_k x(k)h(n-k)}^{y(n)} e^{-j\omega n} \cdot e^{jwk} \cdot e^{-jwk}$$

$$= \sum_n \sum_k x(k)h(n-k)e^{-j\omega(n-k)} e^{-j\omega k}$$

$$= \sum_k x(k)e^{-j\omega k} \sum_n h(n-k)e^{-j\omega(n-k)}$$

$$= H(\omega)X(\omega)$$

Appendix

Show the Fourier transform

$$x_{\text{inverse}}(n) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \, d\omega$$

$$X(\omega) \overset{\text{def}}{=} \sum_k x(k) e^{-j\omega k}$$

$$= \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \overbrace{\sum_{k=-\infty}^{\infty} x(k) e^{-j\omega \cancel{T} k}}^{X(\omega)} \cdot e^{+j\omega n} \, d\omega$$

$$= \frac{1}{2\pi} \cdot \sum_{k=-\infty}^{\infty} x(k) \cdot \int_{-\pi}^{\pi} e^{-j\omega \cancel{T} k} \cdot e^{+j\omega n} \, d\omega$$

$$= \frac{1}{2\pi} \cdot \sum_{k=-\infty}^{\infty} x(k) \cdot \underbrace{\int_{-\pi}^{\pi} e^{-j\omega \cancel{T}(k-n)} \, d\omega}_{2\pi \text{ if } k = n, \text{ 0 otherwise}} = x(n)$$

Show the Fourier transform

Given:

$$H(\omega) = \frac{2e^{j\omega}}{1 - 0.5e^{-j\omega}}$$

Show:

$$\frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} H(\omega)\,d\omega = 1$$

Inv. Fourier Tr

$$\Rightarrow h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n}\, d\omega \Rightarrow h(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)\, d\omega$$

Show the Fourier transform

Given:

$$H(\omega) = \frac{2e^{j\omega}}{1 - 0.5e^{-j\omega}}$$

Show:

$$\frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} H(\omega)\, d\omega = 1$$

$$\Rightarrow H(z) = \frac{2 \cdot z}{1 - 0.5z^{-1}} = 2 \cdot z \cdot \underbrace{\frac{1}{1 - 0.5z^{-1}}}_{(1/2)^n u(n)}$$

$$\Rightarrow h(n) = 2\left(\frac{1}{2}\right)^{n+1} u(n+1)$$

$$h(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega)\, d\omega = 2 \cdot \frac{1}{2} = 1$$

Show the Fourier transform numerically

In Matlab:

```
>> N = 1001;
>> w = linspace(-pi, pi, N);
>> dw = 2*pi/(N-1);
>> H = 2*exp(1j*w) ./ (1-0.5*exp(-1j*w));
>> sum(trapz(H)*dw/2/pi)
ans =
    1.0000 - 0.0000i
```

$$\left(H(\omega) = \frac{2e^{j\omega}}{1 - 0.5e^{-j\omega}} \right)$$

$$\frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} H(\omega)\, d\omega$$