Lecture 6

Digital Signal Processing

Chapter 4

Fourier transform of analog and digital signals

Filtering

Input-output relations:

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n)$$

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = X(f)H(f)$$

We can use the convolution operation to determine the output signal.

$$y(n) = h(n) * x(n) = \sum_{k} x(k)h(n-k)$$

We often classify filters according to the characteristic of $H(\omega)$.

	Filter type	Description
LP	Low pass filter	Passes low frequencies and stops high frequencies.
HP	High pass filter	Passes high frequencies and stops low frequencies.
BP	Band pass filter	Passes a limited frequency band.
BS	Band stop filter	Stops a limited frequency band.

Relation to the z-transform

if h(n) causal, i.e.

$$H(\omega) = \sum_{n=0}^{\infty} h(n) e^{-j\omega n}$$

$$h(n) = 0 \text{ for } n < 0$$

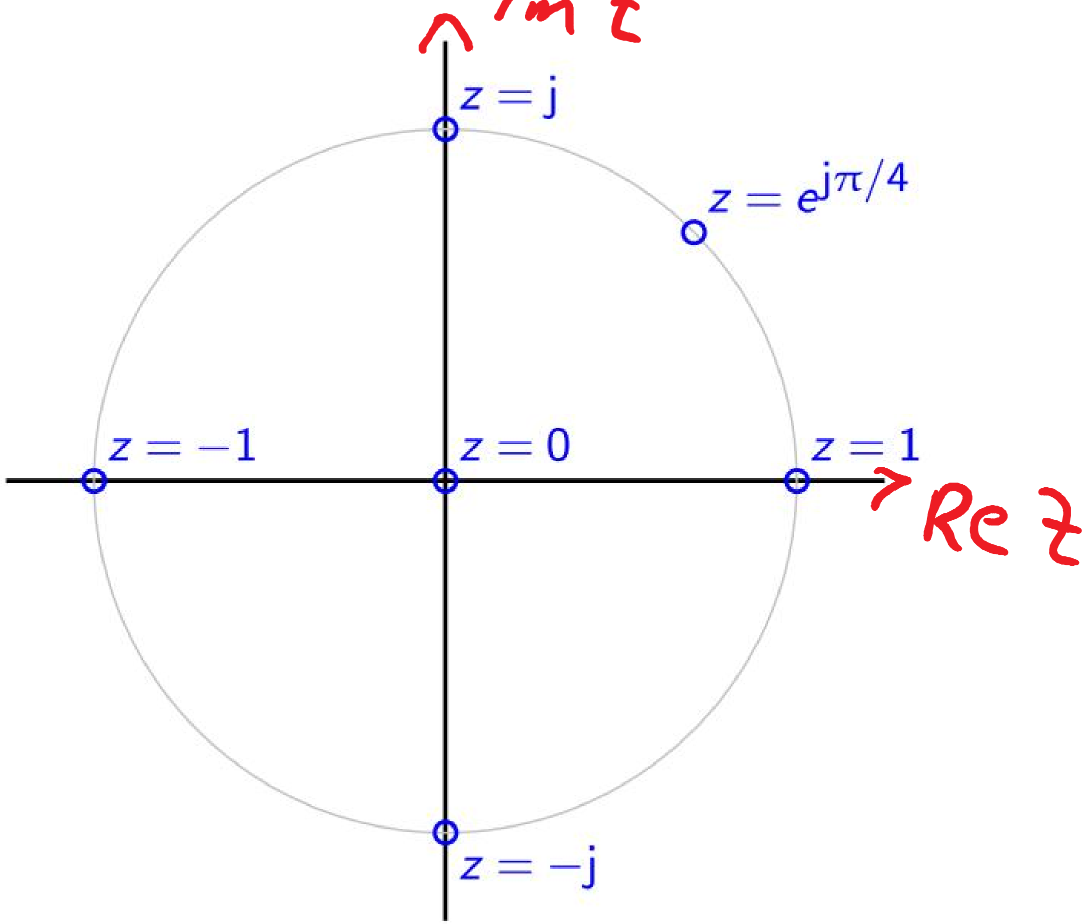
$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

The Fourier transform is the z-transform evaluated on the unit circle if $h(n)$ is causal and stable

$$\boxed{H(\omega) = H(z \mid z = e^{j\omega})}$$

$$z \to e^{j2\pi f}$$

The unit circle is the frequency axis in the discrete domain.



Point in the z-plane	Frequency	
$z = 1$	$f = 0$	$\omega = 0$
$z = \mathrm{j}$	$f = 0.25$	$\omega = \pi/2$
$z = -1$	$f = \pm 0.5$	$\omega = \pm\pi$
$z = -\mathrm{j}$	$f = -0.25$	$\omega = -\pi/2$
$z = e^{\mathrm{j}\pi/4}$	$f = 0.125$	$\omega = \pi/4$

$w = 2\pi f$

Expanding the Fourier transform

Cosine (unstable signal):

$$x(n) = \cos(\omega_0 n) = \frac{1}{2} \cdot \left(\mathrm{e}^{\mathrm{j}\omega_0 n} + \mathrm{e}^{-\mathrm{j}\omega_0 n}\right)$$

$$X(\omega) = \frac{1}{2} \cdot [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \qquad -\pi < \omega < \pi$$

Sine (unstable signal):

$$x(n) = \sin(\omega_0 n) = \frac{1}{2\mathrm{j}} \cdot \left(\mathrm{e}^{\mathrm{j}\omega_0 n} - \mathrm{e}^{-\mathrm{j}\omega_0 n}\right)$$

$$X(\omega) = \frac{1}{2\mathrm{j}} \cdot [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] \qquad -\pi < \omega < \pi$$

Step:

$$x(n) = u(n)$$

$$\Rightarrow X(z) = \frac{1}{1-z^{-1}}$$

$$X(\omega) = \frac{1}{1-e^{-j\omega}} + \frac{1}{2} \cdot \delta(\omega) \qquad -\pi < \omega < \pi$$

Filtering with ideal low pass filter

We want to construct a filter that removes the high frequencies and keeps only the low frequencies.

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n) = \sum_k x(k)h(n-k)$$

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = X(f)H(f) = H(\omega)X(\omega)$$

An ideal low pass filter (non-causal) is defined as

$$H_{\text{ideal}}(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise.} \end{cases}$$

Inverse Fourier transform of a rectangular pulse

$$
\begin{aligned}
h(n) &= \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} H(\omega) \mathrm{e}^{\mathrm{j}\omega n} \, \mathrm{d}\omega \\
&= \frac{1}{2\pi} \cdot \int_{-\omega_c}^{\omega_c} \overbrace{H(\omega)}^{=1} \mathrm{e}^{\mathrm{j}\omega n} \, \mathrm{d}\omega \\
&= \frac{1}{2\pi} \cdot \frac{\mathrm{e}^{\mathrm{j}\omega_c n} - \mathrm{e}^{-\mathrm{j}\omega_c n}}{\mathrm{j}n} \\
&= \frac{\omega_c}{\pi} \cdot \frac{\sin(\omega_c n)}{\omega_c n} \\
&= \frac{\omega_c}{\pi} \cdot \mathrm{sinc}(\omega_c n)
\end{aligned}
$$

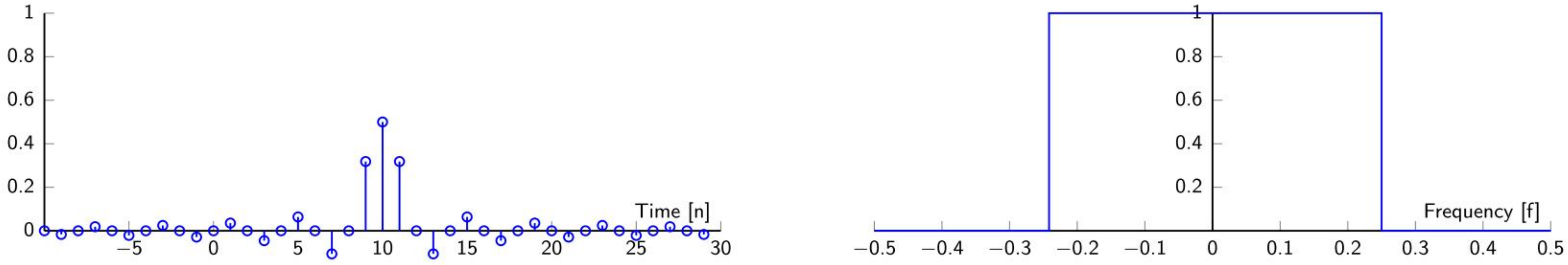
Non-causal and infinite length

Truncation of an ideal low pass filter

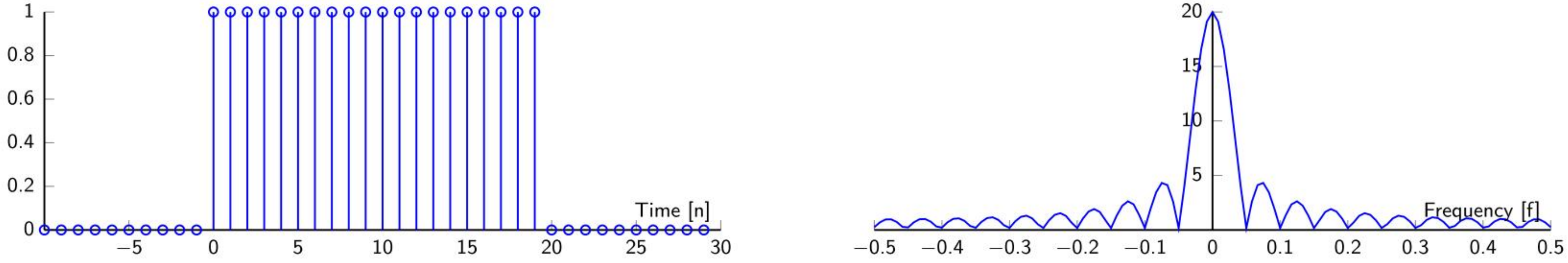
A causal low pass FIR filter can be obtained by selecting N values around the origin and then delay the impulse response by $(N-1)/2$ samples (choose N odd).

$$h(n) = \frac{\omega_c}{\pi} \cdot \frac{\sin\left(\omega_c\left(n - \frac{N-1}{2}\right)\right)}{\omega_c\left(n - \frac{N-1}{2}\right)} \qquad \text{for } 0 \leq n < N \tag{19}$$

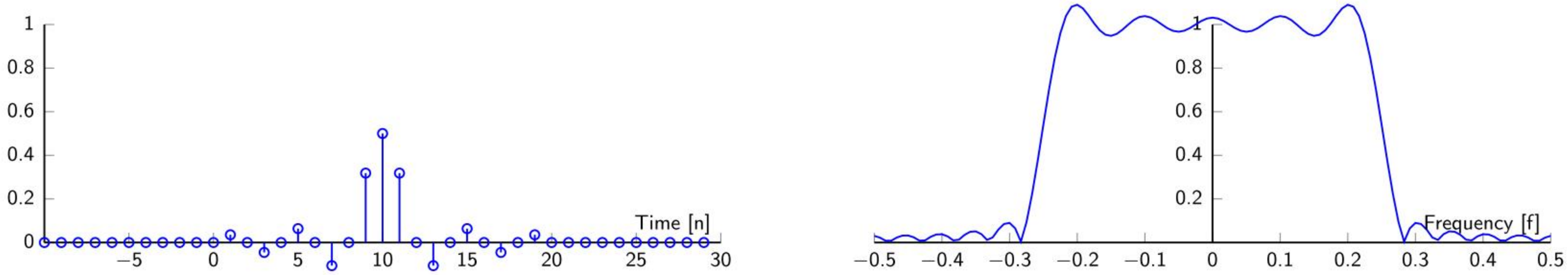
Ideal low pass filter.



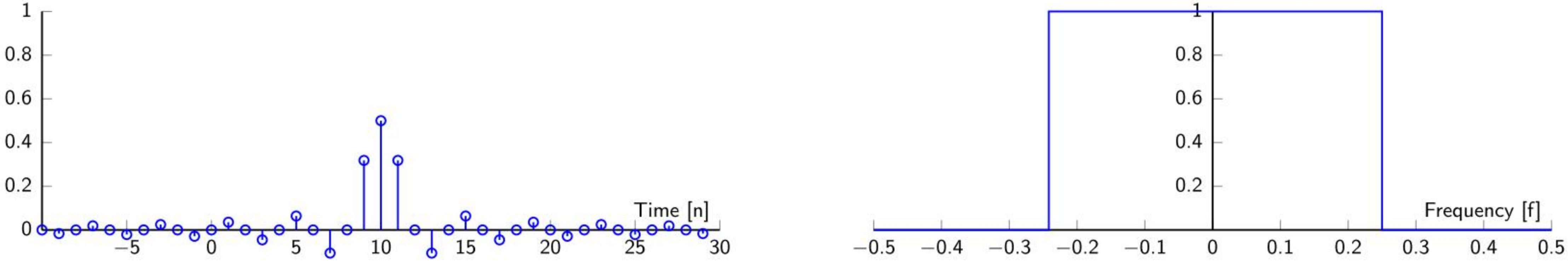
Rectangular window.



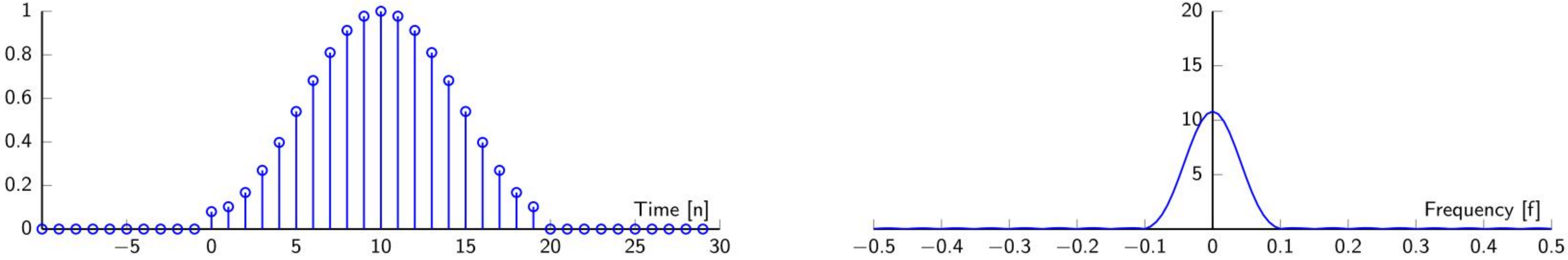
Truncated filter.



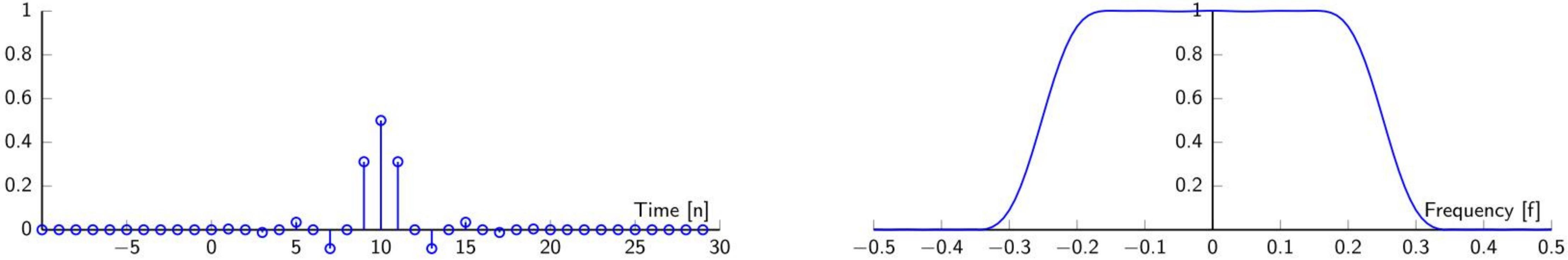
Ideal low pass filter.



Hamming window.



Truncated filter.



Appendix: **Fourier series expansion of periodic signals**

Analog harmonic signals

An analog signal is periodic if

$$x(t) = x(t + T_p) = \{\text{if real-valued } x(t)\} =$$

$$= A_0 + A_1 \cos(\cancel{\omega}_0^{\Omega_0} n + \Theta_1) + A_2 \cos(2\cancel{\omega}_0^{\Omega_0} n + \Theta_2) + \cdots$$

$$\Omega_0 = \frac{2\pi}{T_p}$$

Appendix: **Fourier series expansion of periodic signals**

Analog harmonic signals

An analog signal is periodic if

$$x(t) = x(t + T_p)$$

In general

$$x(t) = \frac{1}{T_p} \cdot \sum_{k=-\infty}^{\infty} X(k \cdot \Omega_0) e^{j\Omega_0 \cdot kt} \quad , \quad \Omega_0 = 2\pi / T_p$$

where

$$X(\Omega) = \int_{\frac{-T_p}{2}}^{\frac{T_p}{2}} x(t) e^{-j\Omega t} \, dt$$

Fourier Transform of one period

Discrete harmonic signals

A discrete signal can also be composed from harmonic components. Periodicity is defined as

$$x(n) = x(n+N) \tag{29}$$

$$x(n) = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k \cdot \omega_0) e^{j\omega_0 \cdot kn}$$

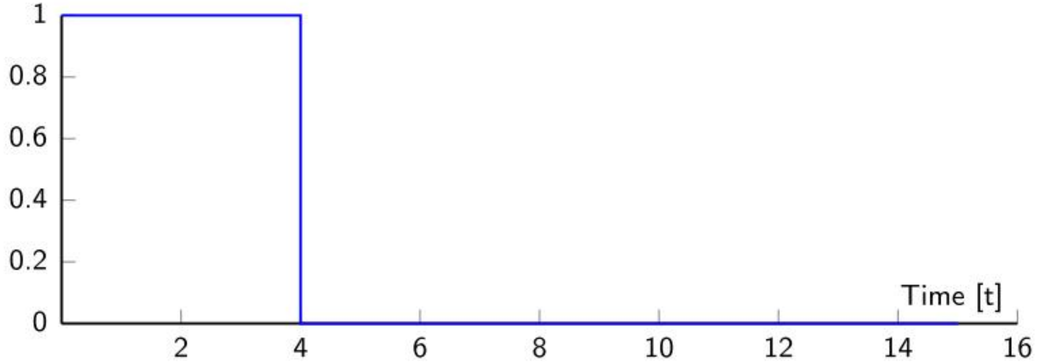
$$\omega_0 = \frac{2\pi}{N}$$

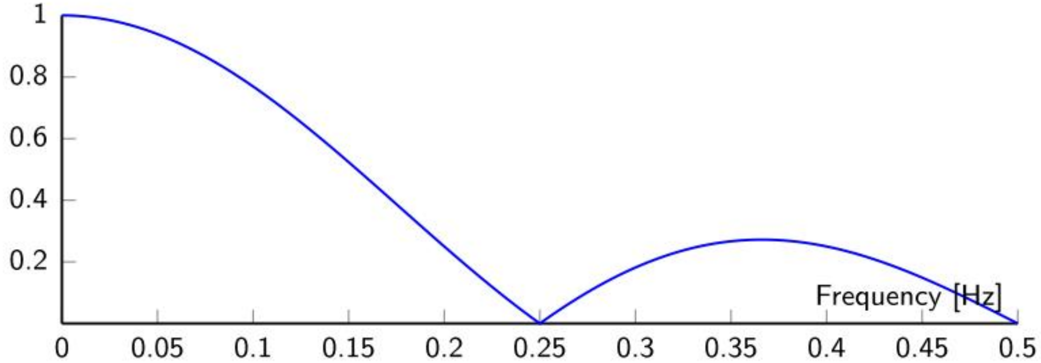
where

$$X(\omega) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n}$$

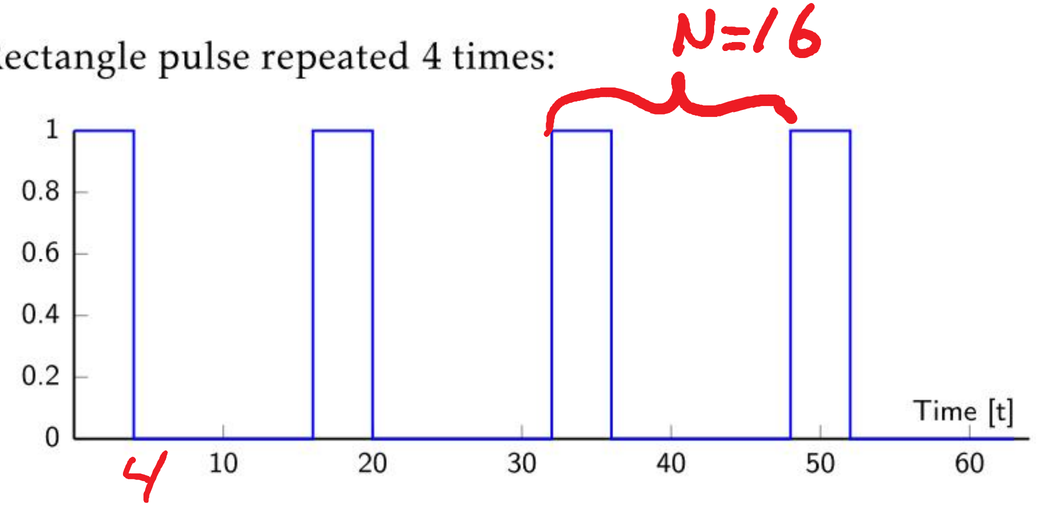
Discrete-Time Fourier Transform of one period

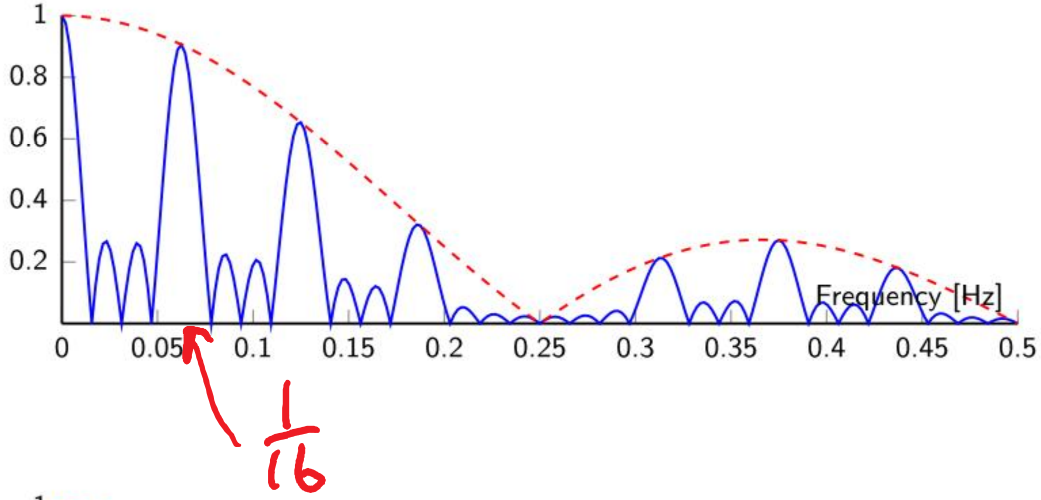
Rectangle pulse:





Rectangle pulse repeated 4 times:





Rectangle pulse repeated 16 times: