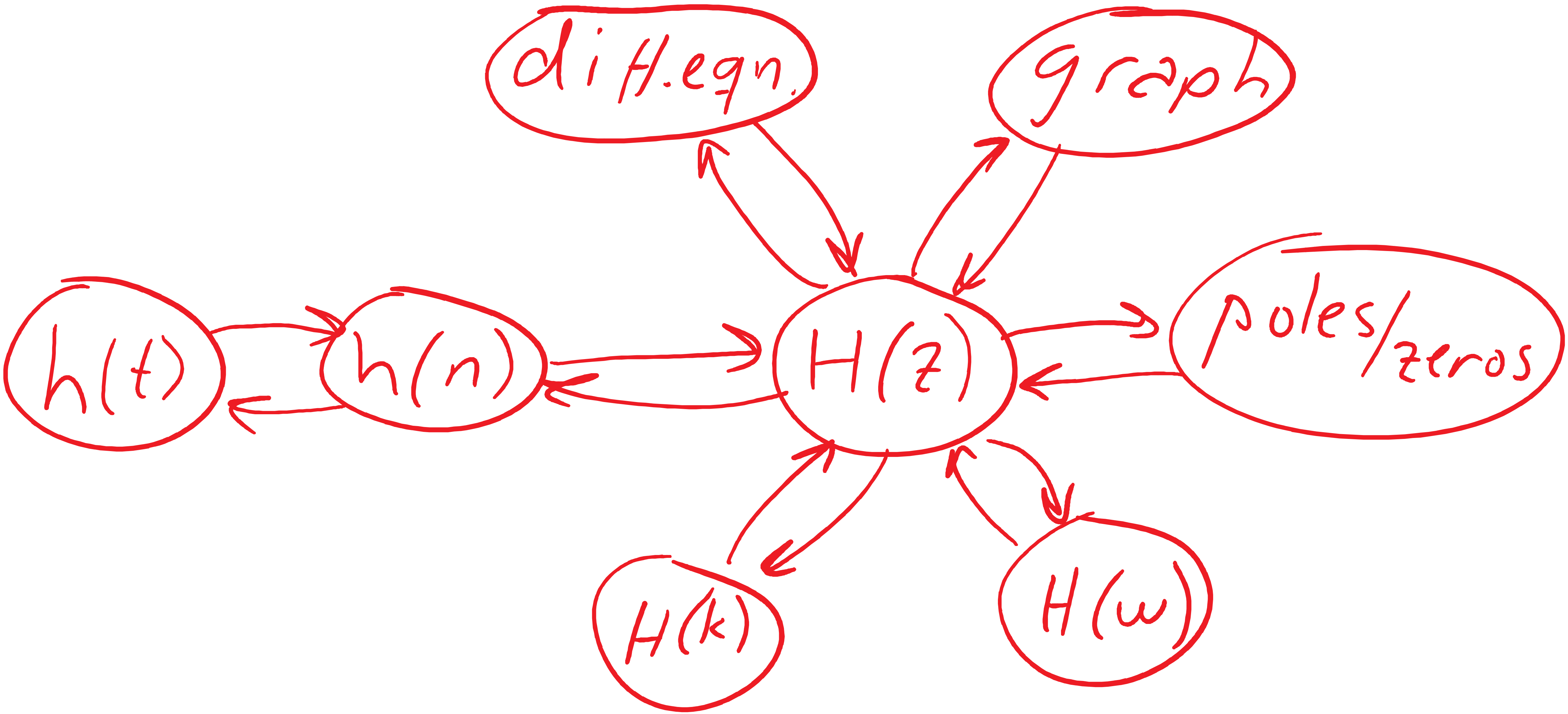
Lecture 7

Digital Signal Processing

Chapter 5

LTI system
Signals in linear systems


diff.eqn.
graph
h(t)
h(n)
H(z)
poles/zeros
H(k)
H(ω)

Linear time invariant systems

Difference equations:

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

The z-transform:

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(\overset{z}{\cancel{n-k}}) \Rightarrow Y(z) = \underbrace{\frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}}_{H(z)} \cdot X(z) = H(z)X(z)$$

Convolution:

$$y(n) = h(n) * x(n)$$

$$= \sum_{k} h(k)x(n-k)$$

We have two kinds of difference equations.

- An FIR system has $a_k = 0$ for all $k \neq 0$. An FIR system therefore has no feedback. The impulse response is $h(n) = \left\{ \begin{array}{cccc} b_0 & b_1 & \cdots & b_M \end{array} \right\}$ which is the same as the coefficients of the difference equation.
- An IIR system has $a_k \neq 0$ for some $k \neq 0$. An IIR system therefore has some feedback.

Fourier transform

If $h(n)$ is causal and stable we have the identity

$$H(\omega) = H(z) \qquad \text{where } z = \mathrm{e}^{\mathrm{j}\omega}$$

and therefore

$$Y(\omega) = \underbrace{\frac{b_0 + b_1 \mathrm{e}^{-\mathrm{j}\omega} + \cdots + b_M \mathrm{e}^{-\mathrm{j}\omega M}}{1 + a_1 \mathrm{e}^{-\mathrm{j}\omega} + \cdots + a_N \mathrm{e}^{-\mathrm{j}\omega N}}}_{H(\omega)} \cdot X(\overset{\omega}{\cancel{z}}) = H(\omega) X(\omega)$$

Sinusoidal signals and LTI systems

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

We want to determine the output signal from an LTI-system, we have two cases;

A) The input signal is causal and the Z-transform exists

We solve this by using the Z-transform, $Y(z) = H(z)X(z)$ and calculate the inverse Z-transform

B) The input signal has (an infinite length) non-causal part

We solve this by using the convolution sum (we cannot use the Z-transform since it does not exist for the input signal)

A) Numerical solution in Matlab

First determine a numerical solution in Matlab.

Given: The input signal

$$x(n) = \cos\left(2\pi \cdot \frac{1}{16} \cdot n\right) \cdot u(n)$$

and the system

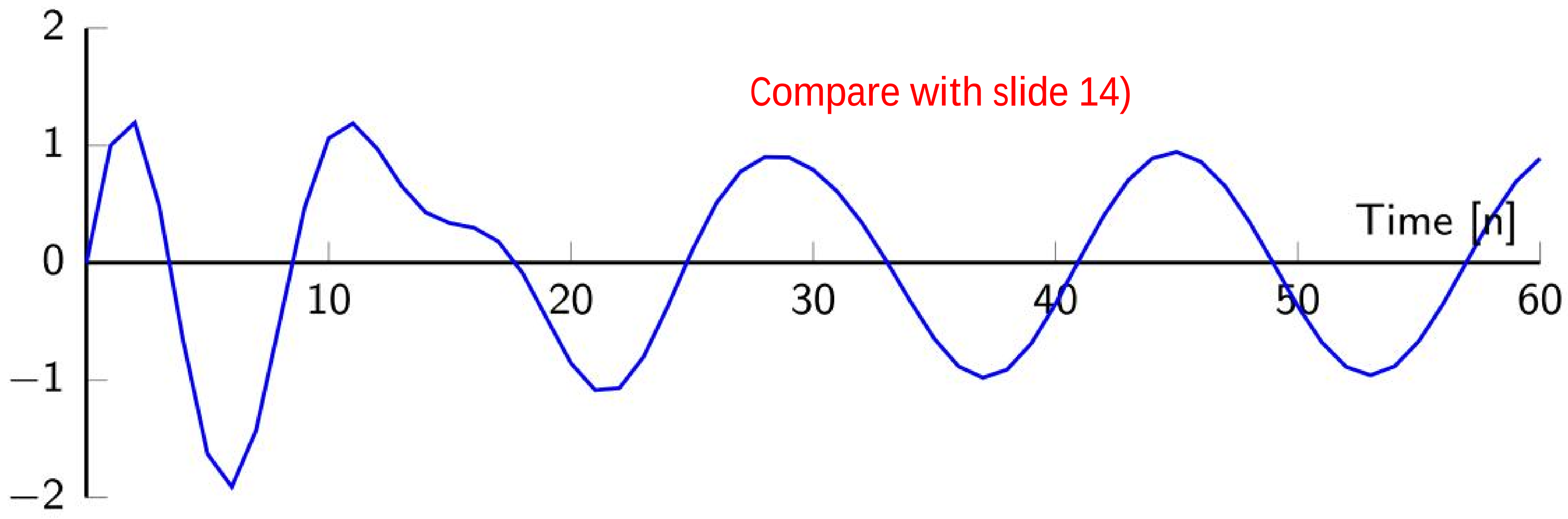
$$H(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}}$$

Find: Determine numerically the output signal $y(n) = x(n) * h(n)$.

```
>> n = 0:60;
>> b = [0, 1, -1];
>> a = [1, -1.27, 0.81];
>> x = cos(2*pi*n/16);
>> y = filter(b, a, x);
>> plot(n, y);
```

we had

$$\boxed{H(z) = \frac{0 + z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}}}$$



We get $y(n)$ = transient solution + stationary solution.

A) Solution using the z-transform

Given: The input signal and the system

$$x(n) = \cos\left(2\pi \cdot \frac{1}{16} \cdot n\right) \cdot u(n) \quad , \qquad H(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}} = \frac{N(z)}{D(z)}$$

notation

This signal is causal and we can determine its z-transform.

$$X(z) = \frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$$

$$\omega_0 = 2\pi \frac{1}{16}$$

$$Y(z) = H(z)X(z) = \frac{N(z)}{D(z)} \cdot \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}} =$$

$$= \left\{\begin{matrix}\text{partial}\\ \text{fraction}\end{matrix}\right\} = \frac{N_1(z)}{D(z)} + \frac{C_0 + C_1 z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

$$\left\{\begin{matrix}\text{Inv. Z-}\\ \text{trans}\end{matrix}\right\} y(n) = \overbrace{\text{transient solution}} + \underbrace{\overbrace{A\cos(\omega_0 n) + B\sin(\omega_0 n)}}_{\text{stationary}}$$

If we want the whole solution we have to determine the partial fraction expansions $N_1(z)$ and $N_2(z) = C_0 + C_1 z^{-1}$ and do the inverse z-transforms.

$$Y(z) = \overbrace{\frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}}}^{H(z)} \cdot \overbrace{\frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}}^{X(z)}$$

Use for instance Matlabs function residuez.m

$$= -0.35 \cdot \frac{1 - 4.177z^{-1}}{1 - 1.27z^{-1} + 0.81z^{-2}} + 0.35 \cdot \frac{1 - 1.896z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$$

$$= -0.35 \cdot \frac{1 - 0.9\cos\left(\frac{\pi}{4}\right)z^{-1}}{1 - 1.27z^{-1} + 0.81z^{-2}} + 0.35 \cdot \frac{5.5629 \cdot \sin\left(\frac{\pi}{4}\right)z^{-1}}{1 - 1.27z^{-1} + 0.81z^{-2}}$$

$$+ 0.35 \cdot \frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}} + 0.35 \cdot \frac{(\cos(\omega_0) - 1.896)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}} \cdot \frac{\sin(\omega_0)}{\sin(\omega_0)}$$

Compare with Lecture 4!!!

Now, using known tables of formulas we get the inverse Z-transform as;

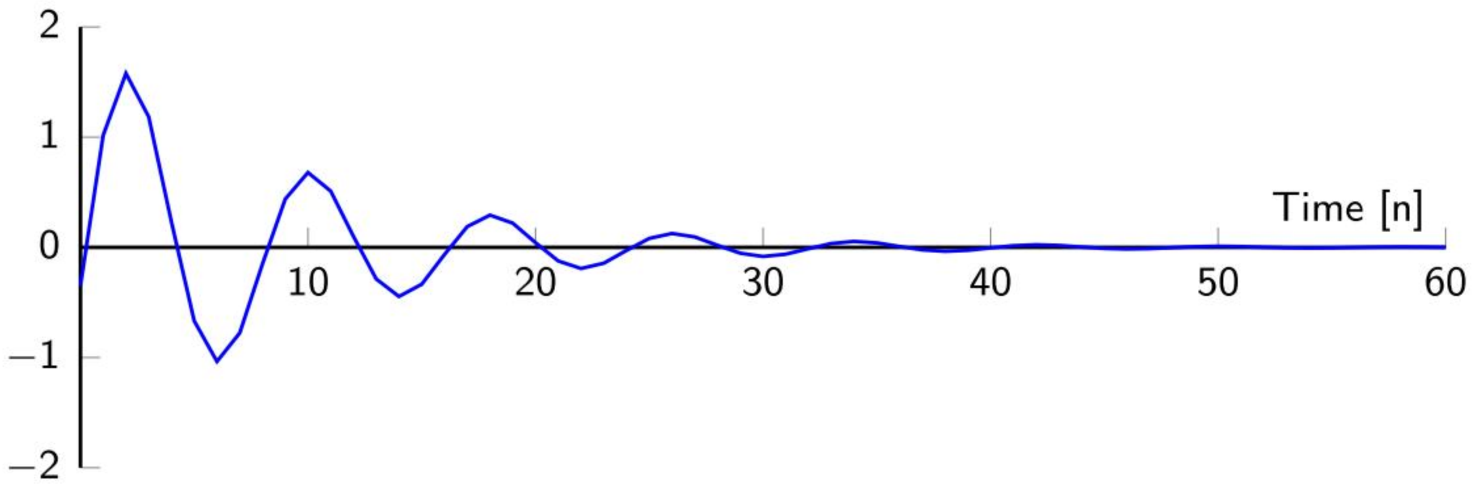
$$y(n) = \underbrace{-0.35 \cdot 0.9^n \cdot \cos\left(2\pi \cdot \frac{1}{8} \cdot n\right) + 0.35 \cdot 5.562 \cdot 0.9^n \sin\left(2\pi \cdot \frac{1}{8} \cdot n\right)}_{\text{Transient}}$$

$$\underbrace{+\,0.35 \cdot \cos(\omega_0 \cdot n) - 0.35 \cdot 2.5392 \cdot \sin(\omega_0 \cdot n)}_{\text{stationary}}$$

Plot the solution in Matlab.

```
>> n = 0:80;
>> yt = -0.35*0.9.^n.*cos(2*pi*n/8) + 0.35*5.562*0.9.^n.*sin(2*pi*n/8);
>> ys = 0.35*cos(2*pi*n/16) - 0.35*2.5392*sin(2*pi*n/16);
>> subplot(3, 1, 1); plot(n, yt);
>> subplot(3, 1, 2); plot(n, ys);
>> subplot(3, 1, 3); plot(n, yt+ys);
```

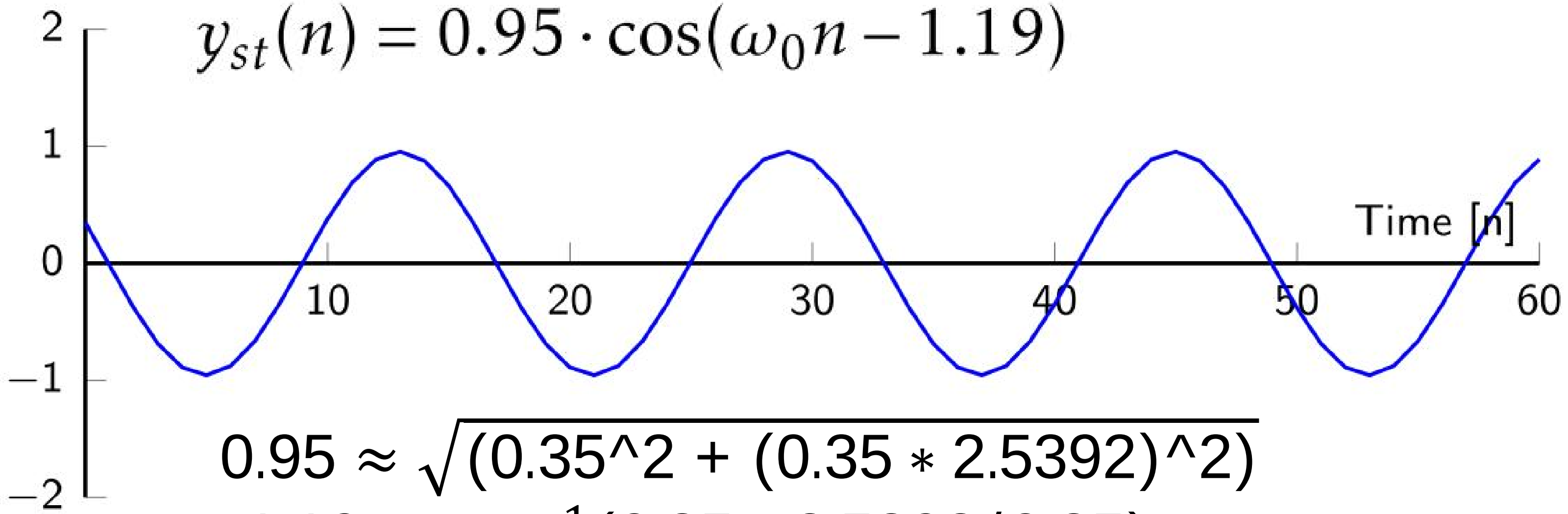
The transient solution:



The stationary solution:

Compare with slide 18)

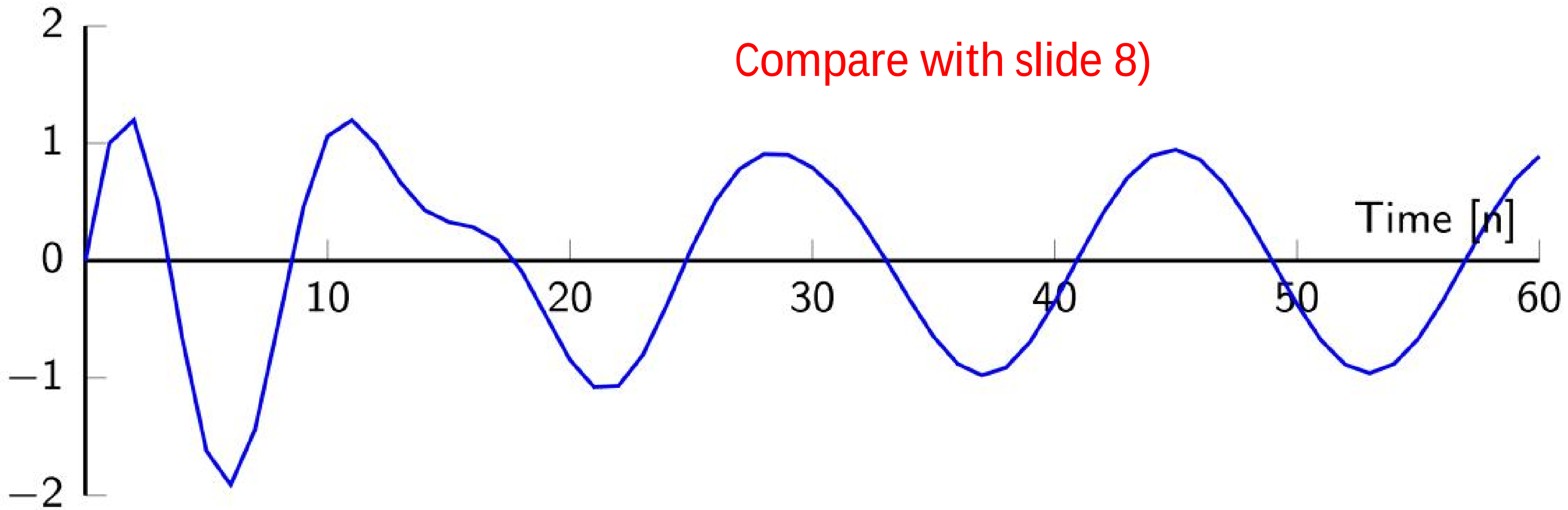
$$y_{st}(n) = 0.95 \cdot \cos(\omega_0 n - 1.19)$$



$$0.95 \approx \sqrt{(0.35\text{^}2 + (0.35 * 2.5392)\text{^}2)}$$

$$1.19 \approx \tan^{-1}(0.35 * 2.5392/0.35)$$

The output signal as the sum of the stationary and the transient solutions.



Compare with slide 8)

We will show that the stationary solution is given by

$$y_{st}(n) = |H(z_0)| \cdot \cos(\omega_0 n + \angle H(z_0)) \quad , \quad \text{where } z_0 = e^{j\omega_0}$$

B) Non-causal input signal of infinite length

Solution without the transient state

The sinusoid is started at $n = -\infty$ and the transient part of the solution has now dissipated.

We start with a complex sinusoidal signal, see page 301-306.

$$x_0(n) = e^{+j\omega_0 n}$$

$$
\begin{aligned}
y_0(n) &= x_0(n) * h(n) \\
&= \sum_{k=-\infty}^{\infty} h(k) x_0(n-k) \\
&= \sum_{k=-\infty}^{\infty} h(k) e^{j\omega_0 (n-k)} \\
&= \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega_0 k} e^{j\omega_0 n} \\
&= H(\omega_0) \cdot e^{j\omega_0 n} = H(\omega_0) \cdot x_0(n)
\end{aligned}
$$

For the whole sinusoidal signal, using both terms of Euler's formula, we get

$$x(n) = \cos(\omega_0 n) = \frac{1}{2} \cdot \left[\mathrm{e}^{\mathrm{j}\omega_0 n} + \cdot \mathrm{e}^{-\mathrm{j}\omega_0 n} \right] = \frac{1}{2} \cdot [x_0(n) + x_0^*(n)]$$

$$y(n) = \frac{1}{2} \cdot \left[H(\omega_0) \cdot \mathrm{e}^{\mathrm{j}\omega_0 n} + H^*(\omega_0) \cdot \mathrm{e}^{-\mathrm{j}\omega_0 n} \right]$$

$$= |H(\omega_0)| \cdot \cos(\omega_0 n + \angle H(\omega_0))$$

In Matlab;

```
>> w0 = 2*pi/16;
>> num = exp(-i*w0) - exp(-i*2*w0);
>> den = 1-1.27*exp(-i*w0)+0.81*exp(-i*2*w0);
>> H0 = num/den;
>> abs(H0), angle(H0)

ans =
      0.9546
ans =
      1.1956
```

$$y_{st}(n) = 0.95 \cdot \cos(\omega_0 n - 1.19)$$

Recall, we had;

$$\boxed{H(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}}}$$

$z \rightarrow e^{j\omega_0}$

Compare with slide 14)

NOTE: This only applies after any initial conditions have dissipated from the system. For an FIR filter of length L, this is after $L-1$ samples. This is called the stationary solution, or the *steady state* solution.

NOTE: This only applies for sinusoidal signals, or for a composite signal (the sum of two or more sinusoidal signals) by computing the response for each component individually.

Linear phase

We often want a filter with linear phase.

$$x(n) \longrightarrow \boxed{H(\omega) = A(\omega)\mathrm{e}^{\mathrm{j}\Phi(\omega)}} \longrightarrow y(n)$$

$$x(n) = \sin(\omega_0 n)$$

$$\begin{aligned} y(n) &= A(\omega_0)\sin(\omega_0 n + \Phi(\omega_0)) \\ &= A(\omega_0)\sin\left(\omega_0\left(n + \frac{\Phi(\omega_0)}{\omega_0}\right)\right) \end{aligned}$$

If $\Phi(\omega_0)/\omega_0$ is constant for all ω_0, then $\Phi(\omega)$ is a straight line in ω. In other words, the filter has linear phase. A filter with linear phase delays all frequencies by the same amount. The time

$$\tau_g = -\frac{\mathrm{d}\Phi(\omega)}{\mathrm{d}\omega} \tag{41}$$

is called the *group delay.*

Example of a filter with linear phase

Given: The impulse response $h(n) = \left\{ \underset{\substack{\uparrow \\ n=0}}{1} \quad 2 \quad 1 \right\}$.

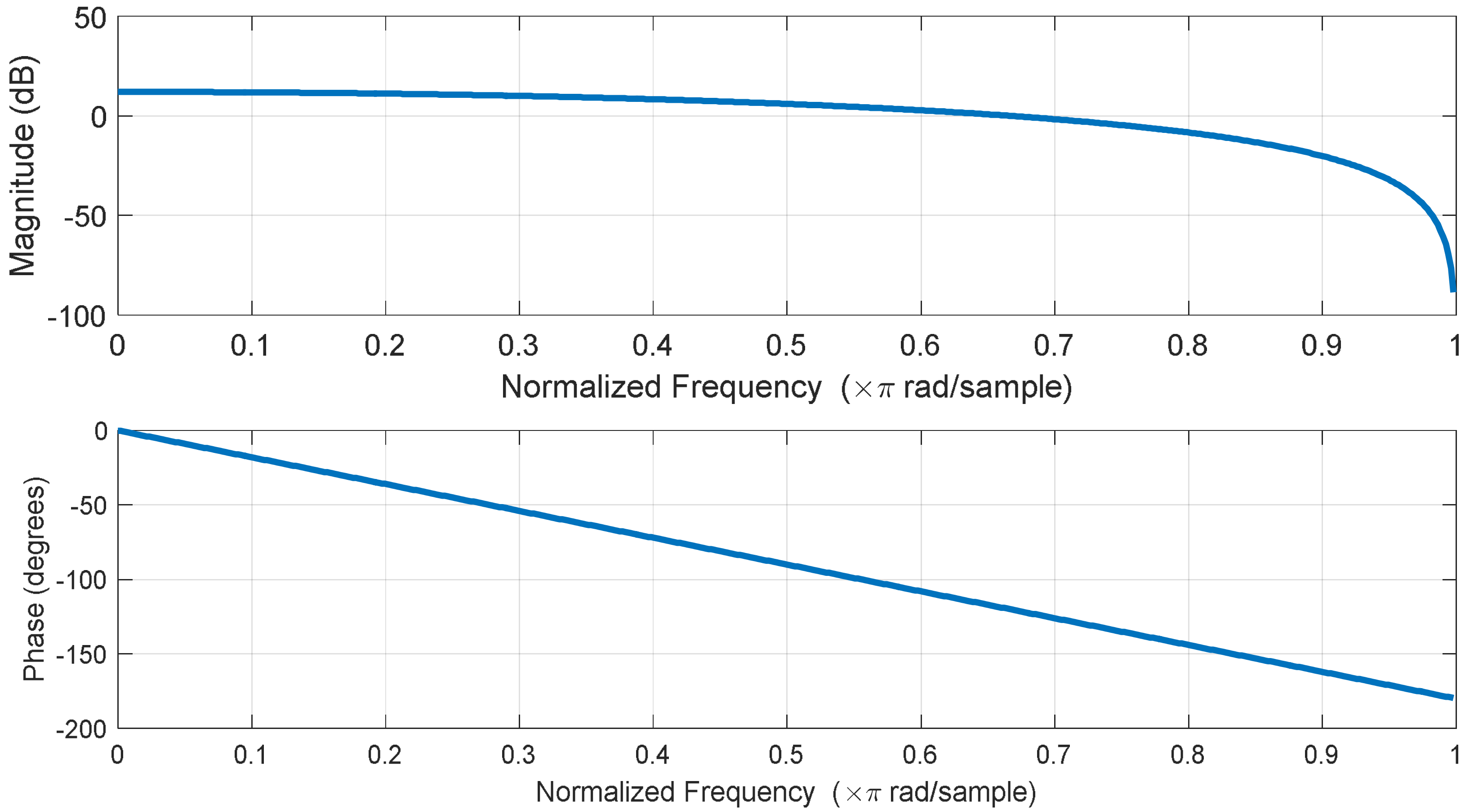
Find: The phase response of $H(\omega)$.

Solution:

$$
\begin{aligned}
H(\omega) &= 1 + 2e^{-j\omega} + e^{-j2\omega} \\
&= e^{-j\omega} \cdot \left(e^{j\omega} + 2 + e^{-j\omega}\right) \\
&= e^{-j\omega} \cdot \underbrace{(2 + 2\cos(\omega))}_{A(\omega)} \\
&= A(\omega) \cdot e^{+j\Phi(\omega)}
\end{aligned}
$$

$$\Phi(\omega) = -\omega$$

In Matlab:
freqz([1 2 1],1)



In Maltlab:
freqz([1 2 2],1)