Lecture 8

Digital Signal Processing

Chapter 5

LTI systems

Convolution and the z-transform

Example E3.3

Given: The system

$$y(n) - y(n-1) + \frac{3}{16} \cdot y(n-2) = x(n)$$

where

$$x(n) = \left(\frac{1}{2}\right)^n \cdot u(n) + \sin\left(2\pi \cdot \frac{1}{4} \cdot n\right)$$

Find: The output signal $y(n)$. Also find the impulse response h(n)!

Solution:

$$H(z) = \frac{1}{1 - z^{-1} + \frac{3}{16} \cdot z^{-2}} = \frac{z^2}{z^2 - z + \frac{3}{16}}$$ $= \frac{z}{(z-p_1)} \cdot \frac{z}{(z-p_2)} = \frac{1}{(1-p_1 z^{-1})} \cdot \frac{1}{(1-p_2 z^{-1})}$

The poles of the filter is given by

$$z^2 - z + \frac{3}{16} = 0 \quad \rightarrow \quad p_1 = 0.25 \quad \text{and} \quad p_2 = 0.75$$

$$\Rightarrow H(z) = \frac{1}{(1-p_1 z^{-1})(1-p_2 z^{-1})} = \frac{-0.5}{1-p_1 z^{-1}} + \frac{1.5}{1-p_2 z^{-1}}$$

$$\Rightarrow h(n) = (-0.5 \cdot 0.25^n + 1.5 \cdot 0.75^n)\, u(n)$$ = Impulse response

Split the input signal into the two terms

$$x(n) = \underbrace{\left(\frac{1}{2}\right)^n \cdot u(n)}_{x_1(n)} + \underbrace{\sin\left(2\pi \cdot \frac{1}{4} \cdot n\right)}_{x_2(n)}$$

First term

$$x_1(n) = 0.5^n \cdot u(n)$$

$$Y_1(z) = \overbrace{\frac{1}{(1-0.25z^{-1})(1-0.75z^{-1})}}^{H(z)} \cdot \overbrace{\frac{1}{1-0.5z^{-1}}}^{X_1(z)} = \left\{\text{partial fraction}\right\} =$$

$$= \frac{\frac{1}{2}}{1-0.25z^{-1}} + \frac{\frac{9}{2}}{1-0.75z^{-1}} - \frac{4}{1-0.5z^{-1}}$$

The inverse transform gives

$$y_1(n) = 0.5 \cdot 0.25^n \cdot u(n) + 4.5 \cdot 0.75^n \cdot u(n) - 4 \cdot 0.5^n \cdot u(n)$$

Second term

$$x_2(n) = \sin(\omega_0 n) \qquad \text{where } \omega_0 = 2\pi \cdot \frac{1}{4}$$

$$y_2(n) = |H(\omega_0)| \cdot \sin(\omega_0 n + \angle H(\omega_0))$$

Calculate

$$H(\omega_0) = H(z \mid z = e^{+j\omega_0}) = \frac{1}{1 - e^{-j\omega_0} + \frac{3}{16} \cdot e^{-j\omega_0 \cdot 2}} = 0.77 \cdot e^{-j0.88}$$

$$\omega_0 = 2\pi \cdot \frac{1}{4}$$

we had

$$\left(H(z) = \frac{1}{1 - z^{-1} + \frac{3}{16} \cdot z^{-2}} \right)$$

which gives

$$y_2(n) = 0.77 \sin\left(2\pi \cdot \frac{1}{4} \cdot n - 0.88\right)$$

First and second part

The final solution is given by the sum of the two individual solutions.

$$
\begin{aligned}
y(n) &= y_1(n) + y_2(n) \\
&= 0.5 \cdot 0.25^n \cdot u(n) + 4.5 \cdot 0.75^n \cdot u(n) - 4 \cdot 0.5^n \cdot u(n) + \\
&\qquad + 0.77 \sin\left(2\pi \cdot \frac{1}{4} \cdot n - 0.88\right)
\end{aligned}
$$

Filters

Suppose that we have a signal that is disturbed by a sinusoidal signal.

$$x(t) = s(t) + \sin(\Omega_0 t) \qquad \text{where } \Omega_0 = 2\pi \cdot 1250 \text{ or } F_0 = 1250\,\text{Hz}.$$

Start by sampling the signal with the sampling frequency $F_s = 10\,000\,\text{Hz}$.

$$x(n) = s(n) + \sin(\omega_0 n)$$

where

$$\omega_0 = 2\pi \cdot \frac{F_0}{F_s} = 2\pi \cdot 0.125 = 2\pi \frac{1}{8}$$

We want to suppress the tonal disturbance so we need to construct a filter which has $H(\omega_0) = 0$

The output is then given by

$$y(n) = \underbrace{h(n) * s(n)}_{\substack{H(\omega)\cdot S(\omega) \\ \approx S(\omega)}} + \underbrace{|H(\omega_0)|}_{=0} \cdot \sin(\omega_0 n + \angle H(\omega_0))$$

for $\omega_0 = 2\pi \frac{1}{8}$

In terms of poles and zeros, the filter must have zeros at

$$n_{1,2} = \mathrm{e}^{\pm \mathrm{j}\omega_0} = \mathrm{e}^{\pm \mathrm{j}2\pi \cdot 0.125}$$

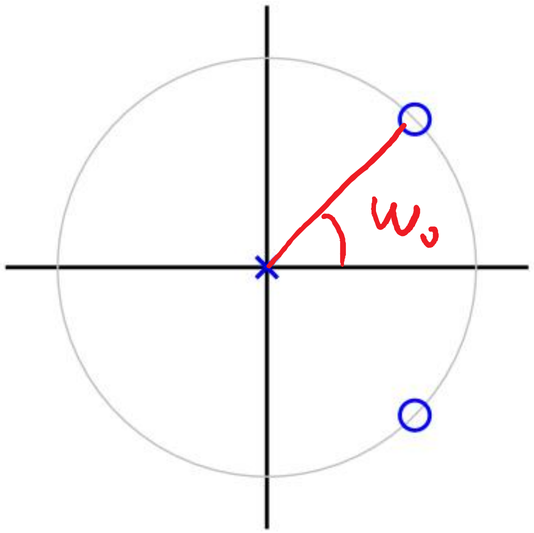
We can choose between an FIR filter or an IIR filter.

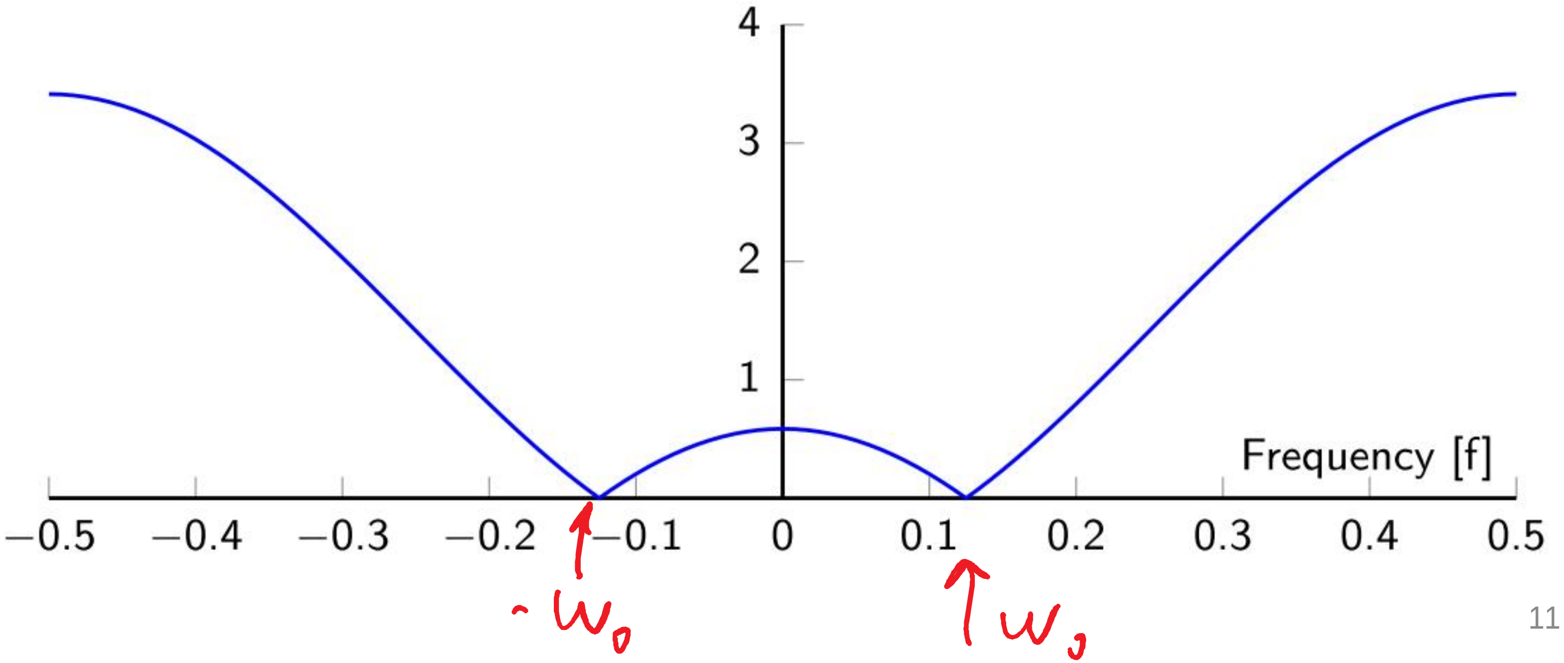
Notch FIR filter

$$H(z) = \frac{\left(z - e^{-j\omega_0}\right)\left(z - e^{j\omega_0}\right)}{z^2} = 1 - 2\cos(\omega_0)z^{-1} + z^{-2}$$

1

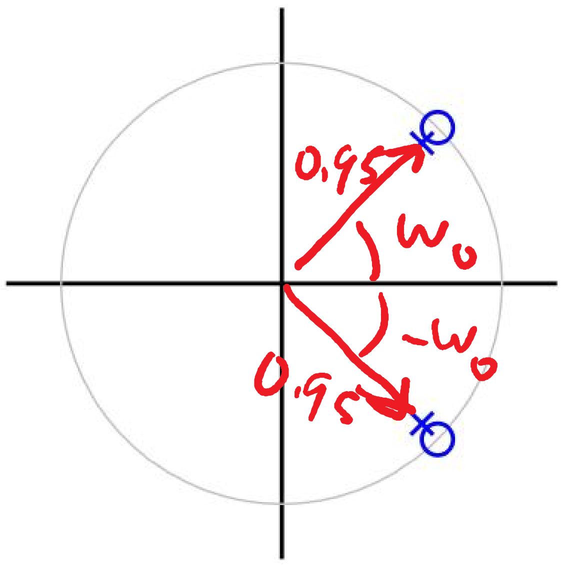
Pole-zero diagram and amplitude response:

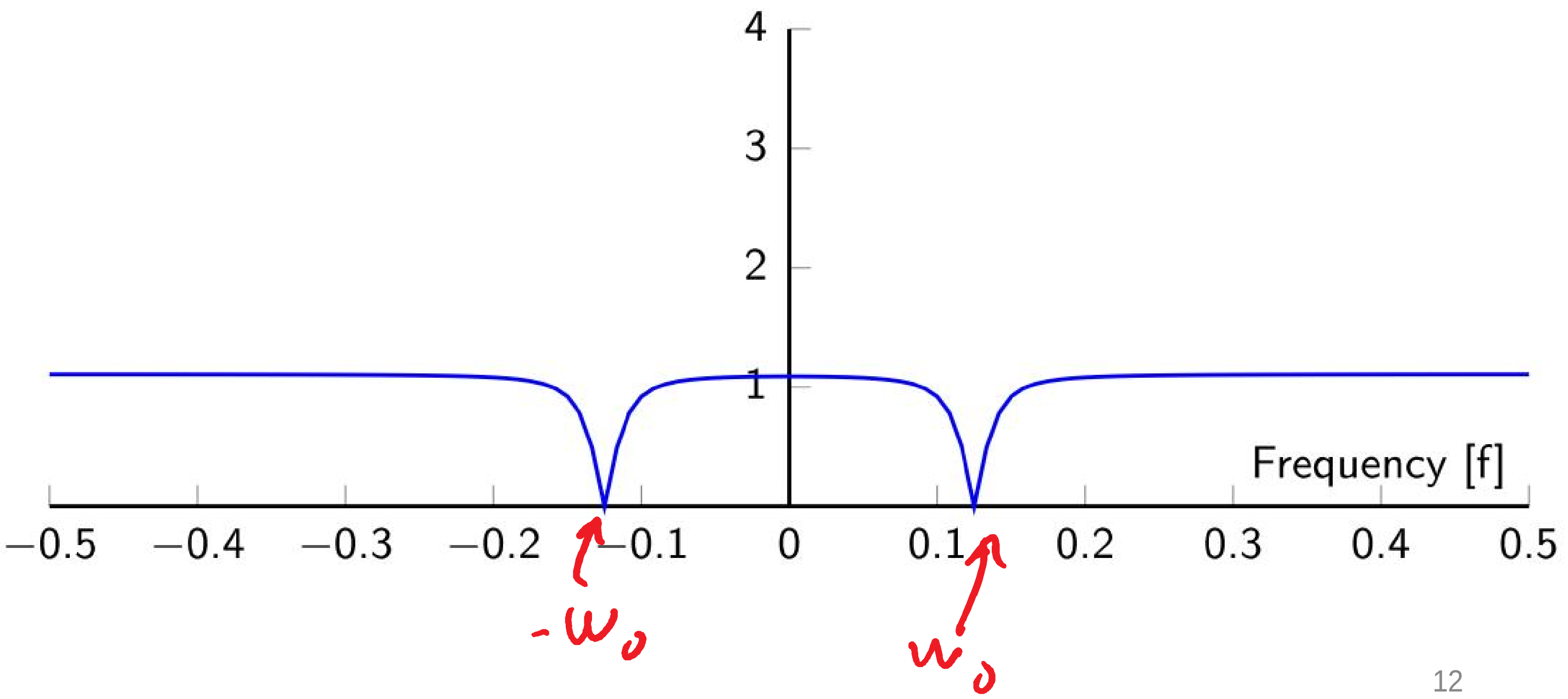




Notch IIR-filter

$$H(z) = \frac{\left(z - e^{-j\omega_0}\right)\left(z - e^{j\omega_0}\right)}{\left(z - \alpha e^{-j\omega_0}\right)\left(z - \alpha e^{j\omega_0}\right)} = \frac{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}{1 - 2 \cdot 0.95 \cdot \cos(\omega_0)z^{-1} + 0.95^2 \cdot z^{-2}}$$





Matlab examples;

```
[x,Fs]=audioread('Speech.wav');
soundsc(x,Fs);
n=0.1*cos(2*pi*1/8*(1:length(x)))';
y=x+n;
soundsc(y,Fs)

p1=0.95*exp(-1i*2*pi*1/8);p2=conj(p1);
n1=exp(-1i*2*pi*1/8);n2=conj(n1);
B=poly([n1,n2]);A=poly([p1,p2]);

z1=filter(B,1,y);figure,zplane(B,1),figure,
freqz(B,1),soundsc(z1,Fs)
z2=filter(B,A,y);figure,zplane(B,A),figure,
freqz(B,A),soundsc(z2,Fs)
```

Linear phase

From Lecture 7: Slide 20

We often want a filter with linear phase.

$$x(n) \longrightarrow \boxed{H(\omega) = A(\omega)e^{j\Phi(\omega)}} \longrightarrow y(n)$$

$$x(n) = \sin(\omega_0 n)$$

$$y(n) = A(\omega_0)\sin(\omega_0 n + \Phi(\omega_0))$$

$$= A(\omega_0)\sin\left(\omega_0\left(n + \frac{\Phi(\omega_0)}{\omega_0}\right)\right)$$

FIR filter with linear phase

An FIR filter with linear phase has a symmetric impulse response.

Symmetry	Description	Filter property
a) $h(n) = h(-n)$	Symmetry around 0.	$H(\omega)$ real. $\angle H(\omega) = \begin{cases} 0 \\ \pm\pi \end{cases}$
b) $h(n) = h(N-n)$	Symmetry around $N/2$.	$H(\omega)$ linear phase.
c) $h(n) = -h(N-n)$	Anti-symmetry around $N/2$.	$H(\omega)$ linear phase.

Example:

a)

- Symmetric impulse response around $n = 0$.

$$h(n) = \{ \; 1 \quad 2 \quad \underline{3} \quad 2 \quad 1 \; \}$$

b)

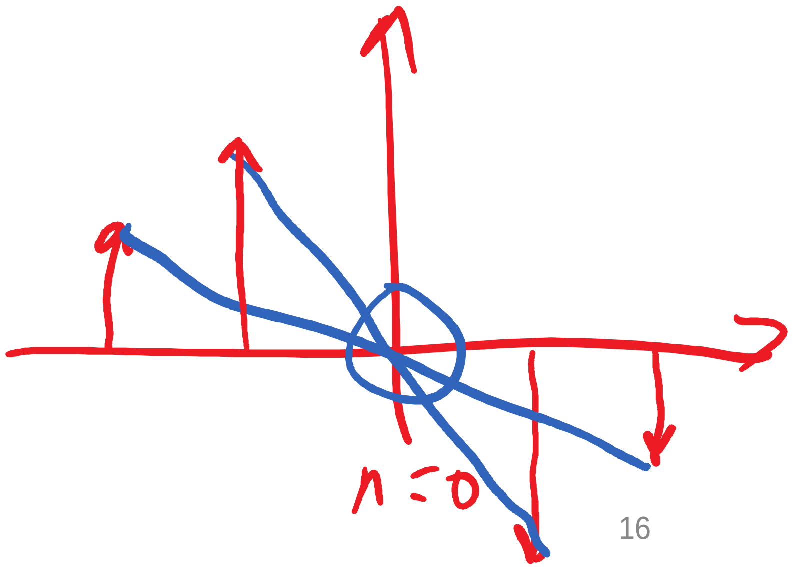
- Causal symmetric impulse response.

$$h(n) = \{ \; \underline{1} \quad 2 \quad 3 \quad 2 \quad 1 \; \}$$

c)

- Causal anti-symmetric impulse response.

$$h(n) = \{ \; \underline{1} \quad 2 \quad 0 \quad -2 \quad -1 \; \}$$



Show that $H(\omega)$ has a linear phase.

$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

$$\begin{aligned}
H(\omega) &= 1 + 2e^{-j\omega} + 3e^{-j2\omega} + 2e^{-j3\omega} + e^{-j4\omega} \\
&= \left(e^{j2\omega} + 2e^{j\omega} + 3 + 2e^{-j\omega} + e^{-j2\omega}\right) \cdot e^{-j2\omega} \\
&= \overbrace{(3 + 4\cos\omega + 2\cos 2\omega)}^{A(w)} \cdot e^{-j2\omega} \\
&= \underbrace{|(3 + 4\cos\omega + 2\cos 2\omega)|}_{k=0 \text{ if } >0 \;\rightarrow\; k=1 \text{ if } <0} \cdot e^{-j2\omega + j\pi\cdot k} \qquad \text{for } k \text{ integer}
\end{aligned}$$

$$\tau_g = -\frac{d(-2w + \pi k)}{dw} = 2$$

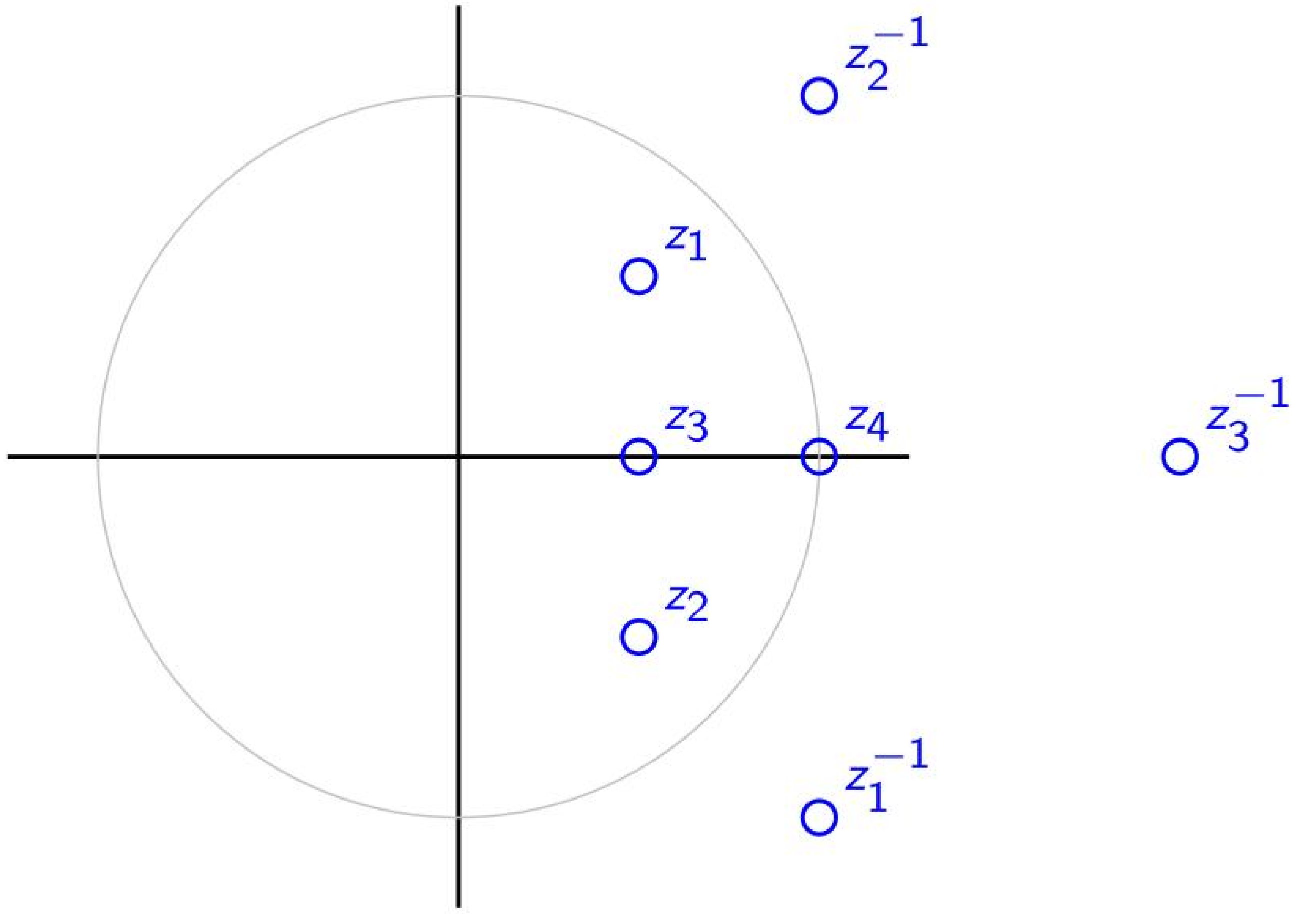
Pole-zero diagram

What does linear phase look like in a pole-zero diagram?

$$
\begin{aligned}
H(z) &= 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4} \\
&= z^{-4} \cdot \left(z^4 + 2z^3 + 3z^2 + 2z + 1\right) \\
&= z^{-4} \cdot H(z^{-1})
\end{aligned}
$$

If z is a zero then z^{-1} must also be a zero.

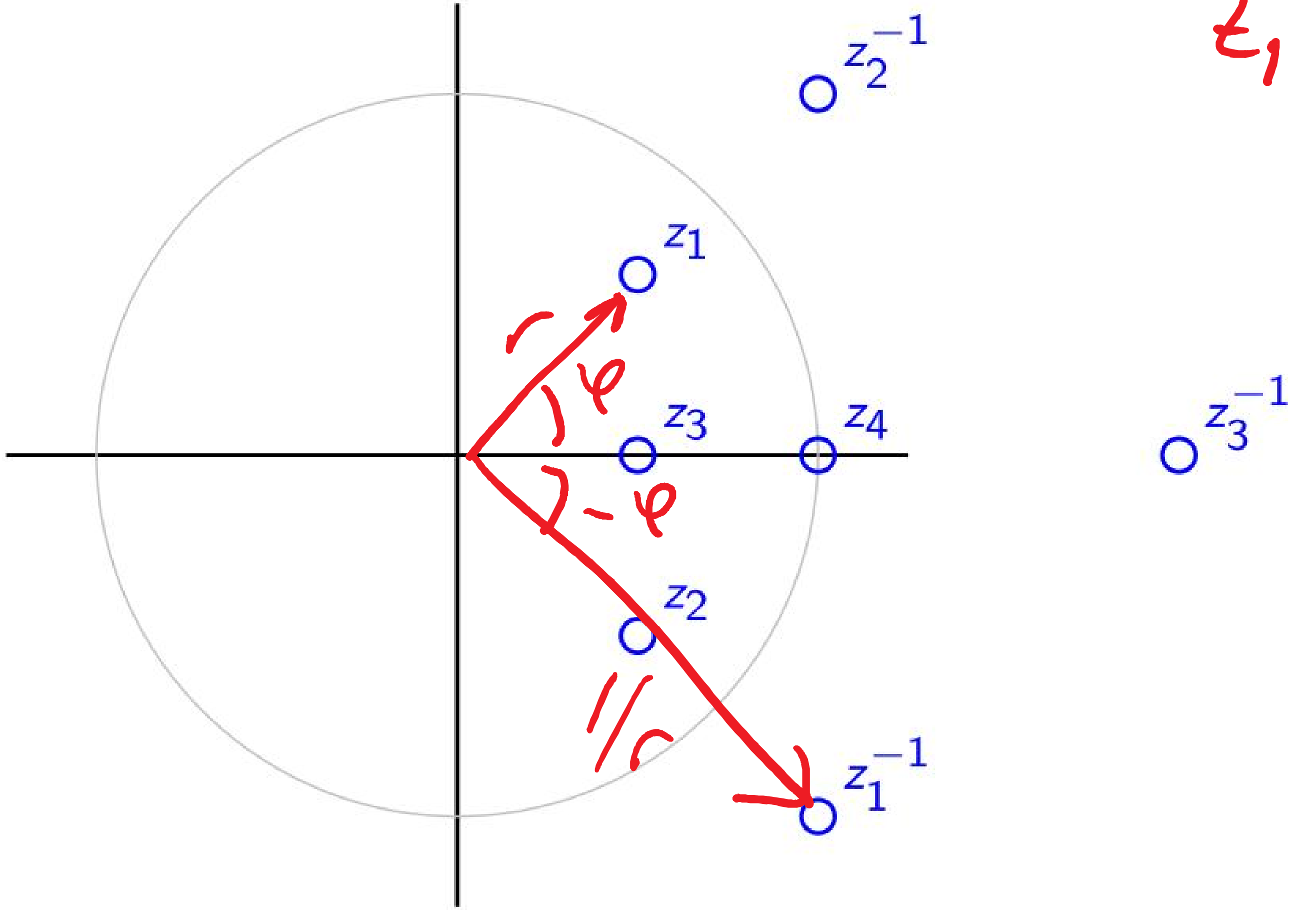
Illustration



Illustration

Ex If $z_1 = re^{j\varphi}$ then

$$\frac{1}{z_1} = \frac{1}{re^{j\varphi}} = \frac{1}{r}e^{-j\varphi}$$

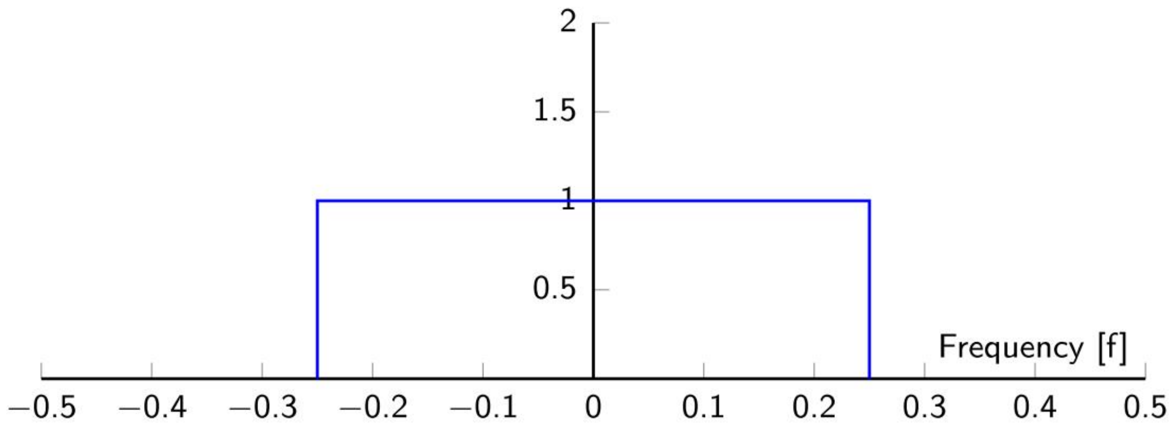


Filter types

Ideal low pass filter

A non-causal ideal low pass filter is defined as

$$H_{\text{ideal}}(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases}$$



The impulse response is

$$h(n) = \frac{\omega_c}{\pi} \cdot \frac{\sin \omega_c n}{\omega_c n}$$

A causal low pass FIR filter can be obtained by selecting N (choose N odd) values around the origin and then delay the impulse response by $(N-1)/2$.

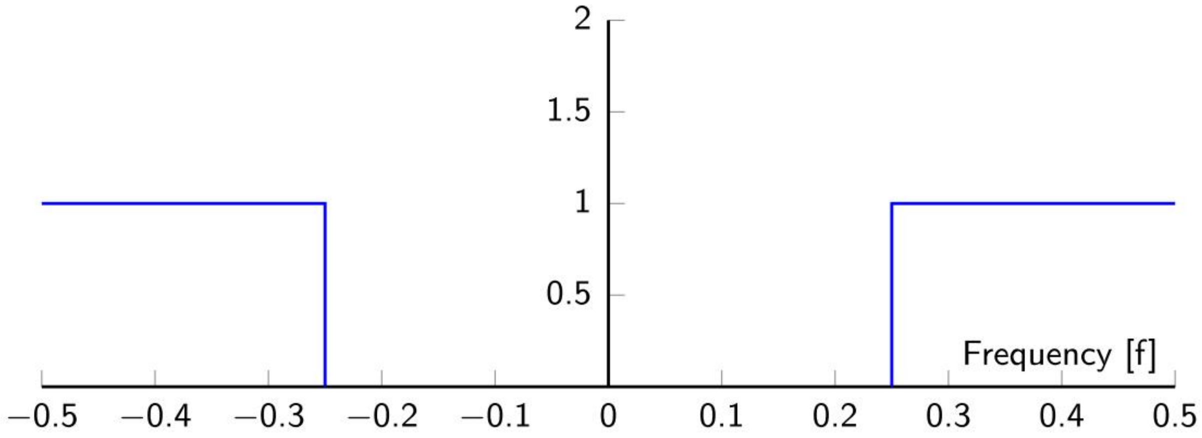
$$h(n) = \frac{\omega_c}{\pi} \cdot \frac{\sin \omega_c \left(n - \frac{N-1}{2}\right)}{\omega_c \left(n - \frac{N-1}{2}\right)} \tag{37}$$

Ideal high pass filter

A non-causal ideal high pass filter is defined as

$$H_{\text{ideal}}(\omega) = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & \text{otherwise} \end{cases}$$

$$= 1 - \text{low pass filter}$$



The impulse response is

$$h(n) = \delta(n) - \frac{\omega_c}{\pi} \cdot \frac{\sin \omega_c n}{\omega_c n}$$

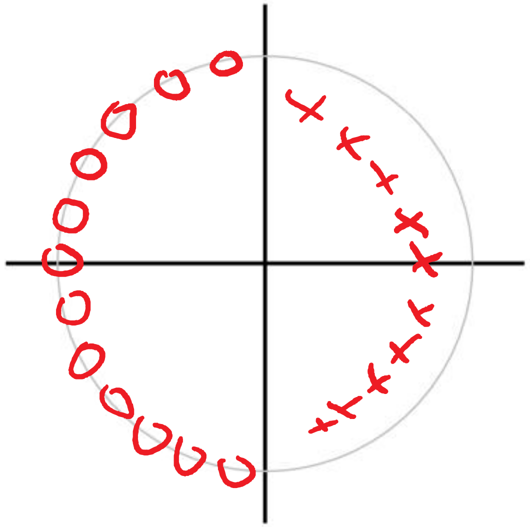
A causal high pass FIR filter can be obtained by selecting N (choose N odd) values around the origin and then delay the impulse response by $(N-1)/2$.

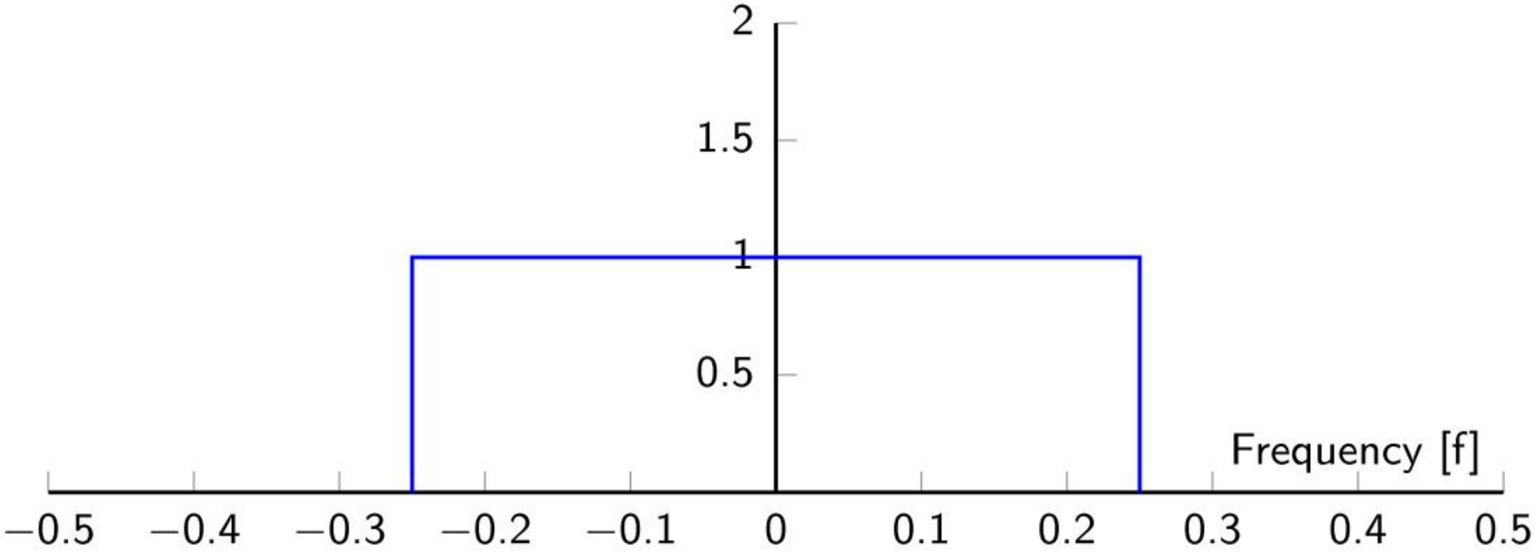
$$h(n) = \delta\left(n - \frac{N-1}{2}\right) - \frac{\omega_c}{\pi} \cdot \frac{\sin \omega_c \left(n - \frac{N-1}{2}\right)}{\omega_c \left(n - \frac{N-1}{2}\right)} \tag{41}$$

Classification of filters

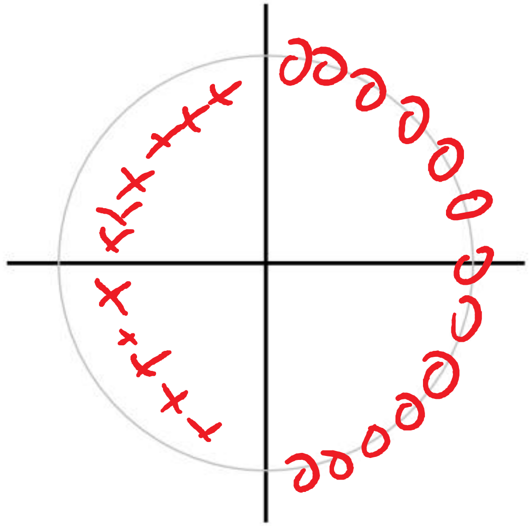
Suggest a pole-zero placement for the filter types (page 330–346).

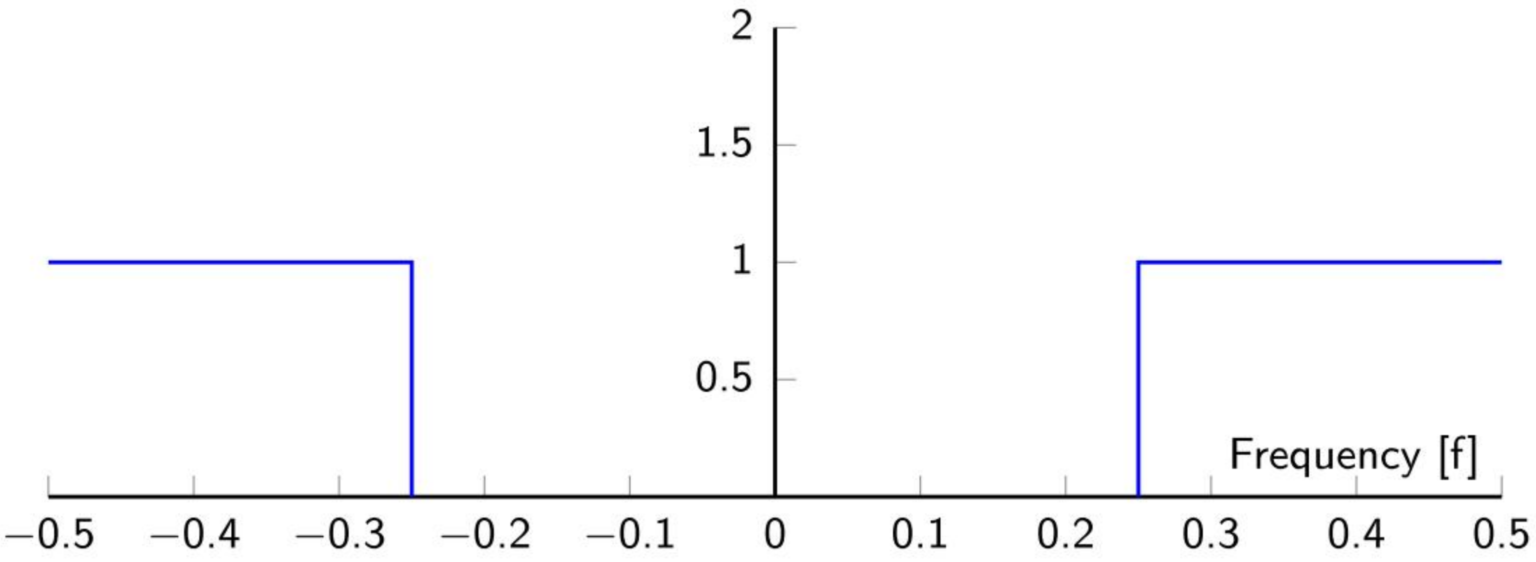
Low pass filter:



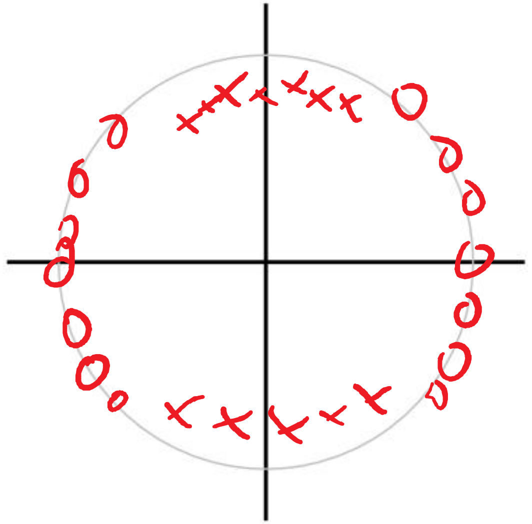


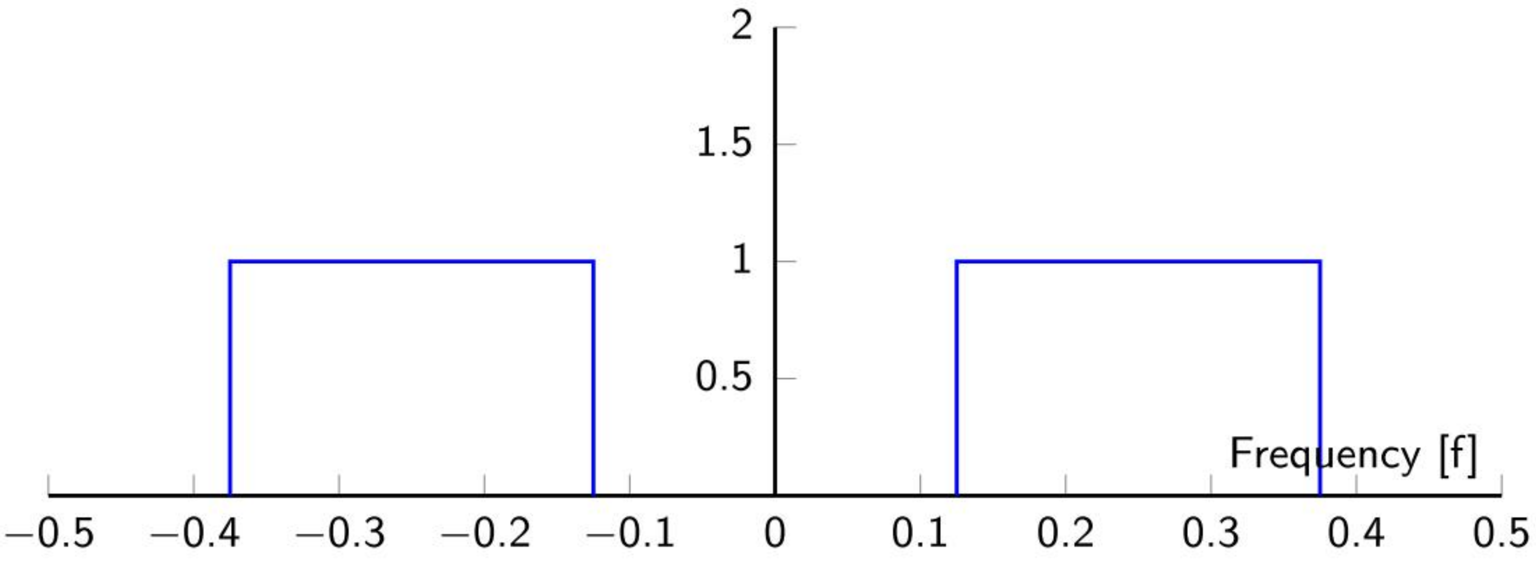
High pass filter:



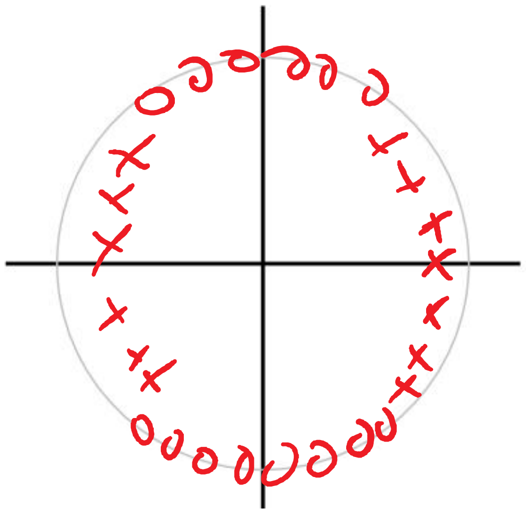


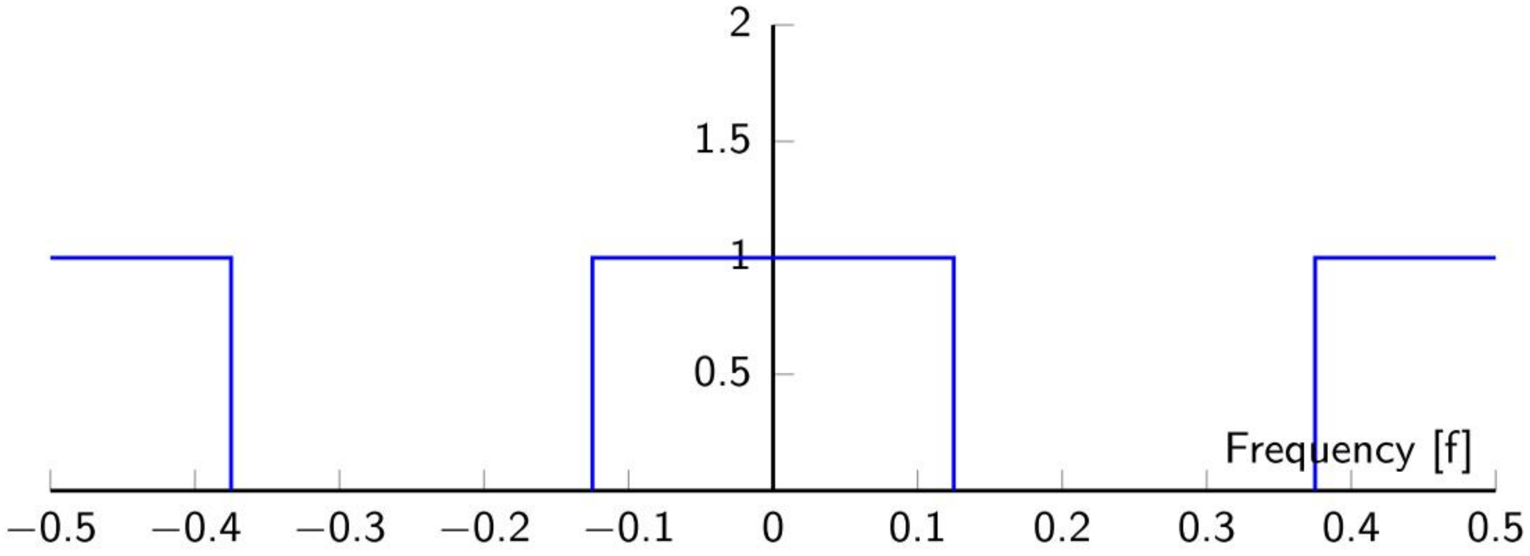
Band pass filter:



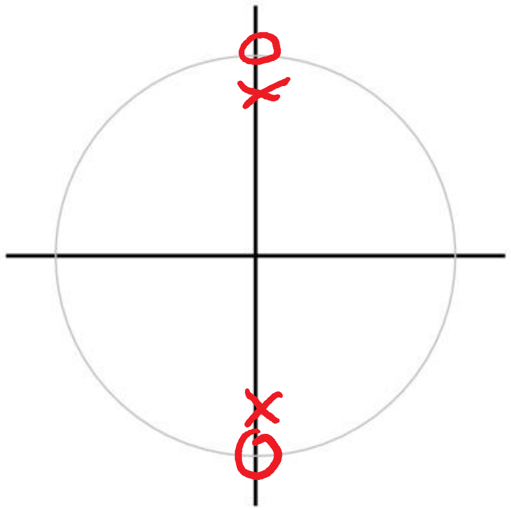


Band stop filter:

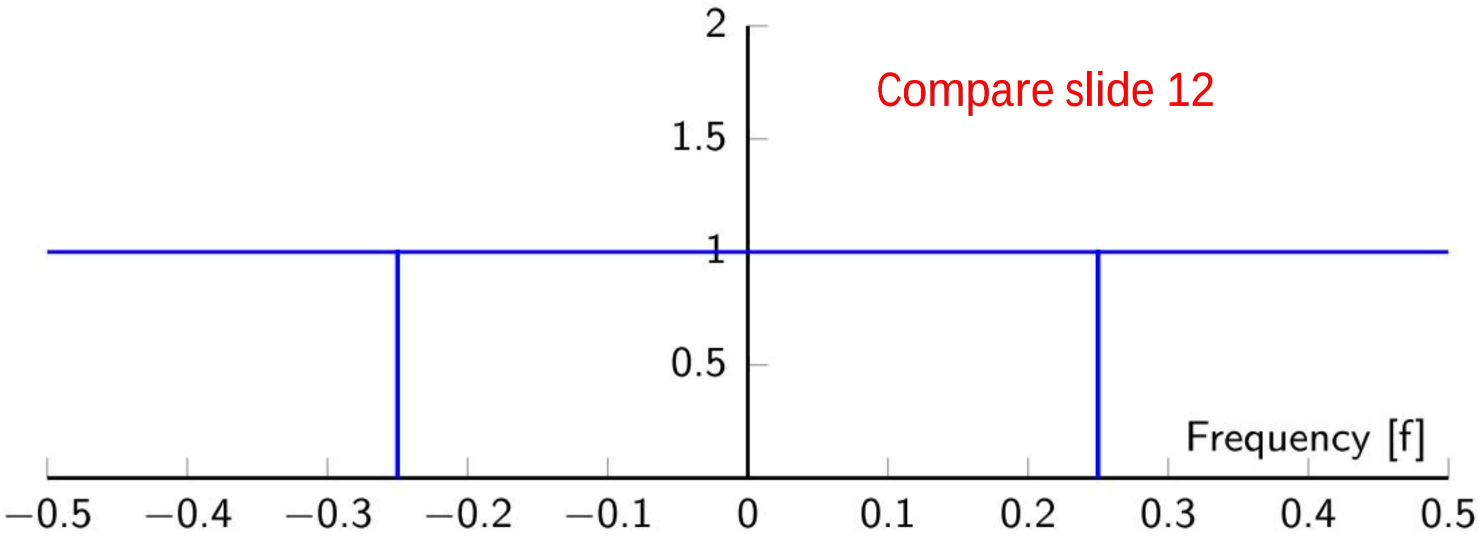




Notch filter:



Compare slide 12



Minimum and maximum phase systems

Def:

A system $H(z)$ with all the zeros inside the unit circle is called a minimum phase system

Def:

Likewise, a system $H(z)$ with all the zeros outside the unit circle is called a maximum phase system

Def:

A system $H(z)$ with zeros both inside and outside the unit circle is called a *mixed phase system*.

We often want minimum or linear phase systems.

Causal FIR and IIR filters

FIR filters:

- The impulse response is of finite duration.
- The filter is always stable.
- All poles are at the origin.
- The filter may have linear phase.

IIR filters:

- The impulse response ~~if~~ is of infinite duration.
- The filter is stable if and only if all poles lie inside the unit circle.
- The filter cannot have linear phase.

Connection between the number of poles N_p, the number of zeros N_z and the impulse response $h(n)$:

Ex

$N_p = N_z$ For example a pole and zero, with the impulse response

$$h(n) = \{ \underline{1} \quad 1 \quad 0 \quad \dots \} \quad \Rightarrow \quad H(z) = 1 + z^{-1} = \frac{z-1}{z}$$

causal

Ex

$N_p = N_z + 1$ For example a zero ~~pole~~ and two poles ~~zeros~~.

$$h(n) = \{\, \underline{0} \quad 1 \quad 1 \quad 0 \quad \ldots \,\} \quad \Rightarrow \quad H(z) = z^{-1} + z^{-2} = \frac{z-1}{z^2}$$

causual

Conclusion:

If the number of poles is greater or equal to the number of zeros, the system is causal. This applies generally for both causal FIR filters and causal IIR filters.