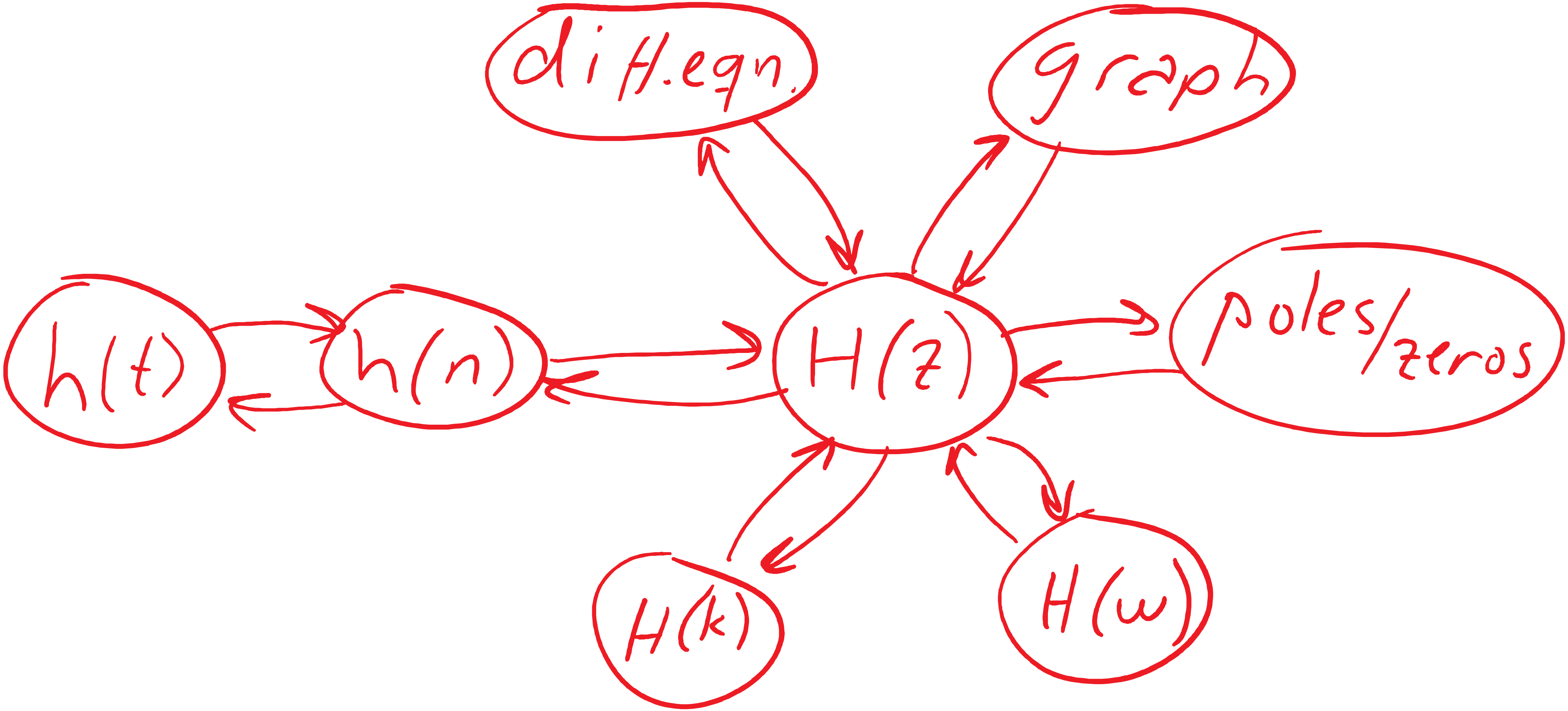
Lecture 9

Digital Signal Processing

Chapter 6

Sampling


diff.eqn.
graph
h(t)
h(n)
H(z)
poles/zeros
H(k)
H(ω)

Sampling

$$x(t) \longrightarrow \underbrace{\boxed{\text{LP}} \longrightarrow \boxed{\text{A/D}}}_{\text{Sampling}} \longrightarrow \underbrace{\boxed{\text{Proc.}}}_{\text{Processing}} \longrightarrow \underbrace{\boxed{\text{D/A}} \longrightarrow \boxed{\text{LP}}}_{\text{Reconstruction}} \longrightarrow y(t)$$

An signal is read at a regular interval

$$t = nT_s = n\frac{1}{F_s}$$

T_s is the time period between each sample and F_s is the sampling rate

T_s [sec] , F_s [Hz]

Some established sample rate standards

TV frame rate	$F_s = 50\,\text{frames/s}$	HD-tv 100/200/400 fr/sec
CD audio	$F_s = 44\,100\,\text{Hz}$	Studio audio up to 48 kHz. DAT
Telephony	$F_s = 8000\,\text{Hz}$	GSM and analog systems.

Sampling

$$x(n) = x(t \mid t = n \cdot \frac{1}{F_s})$$

Example

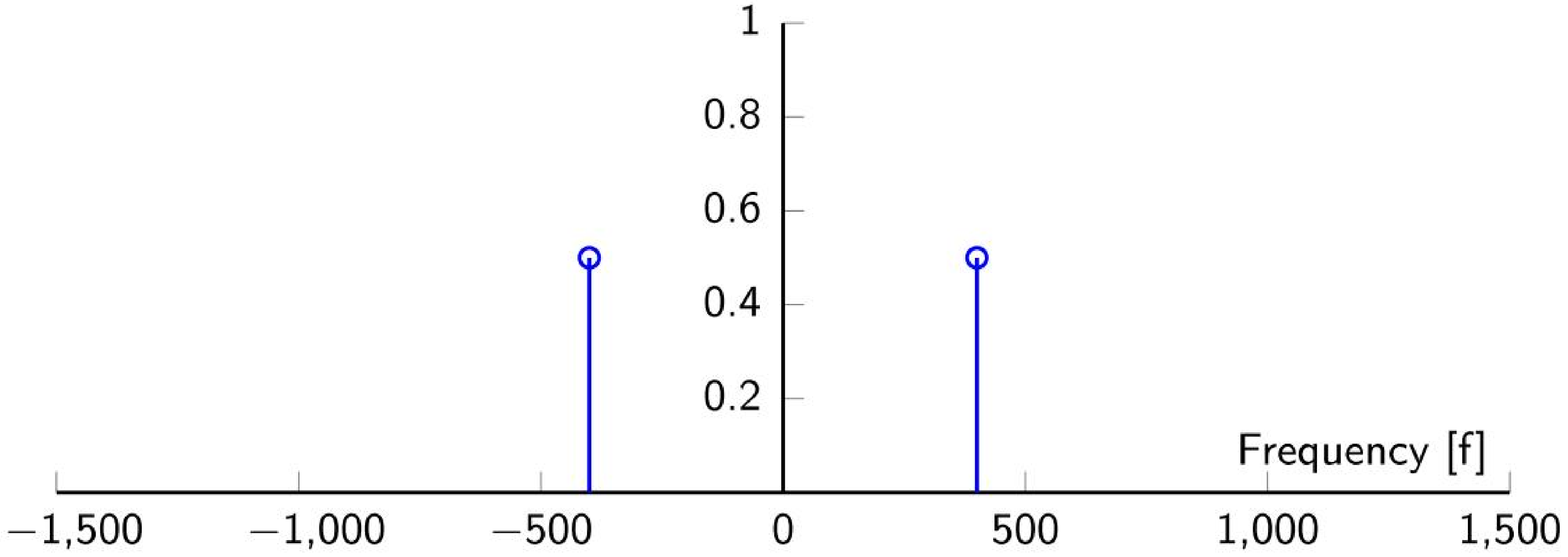
Given:

$$x(t) = \cos(2\pi 400 t)$$

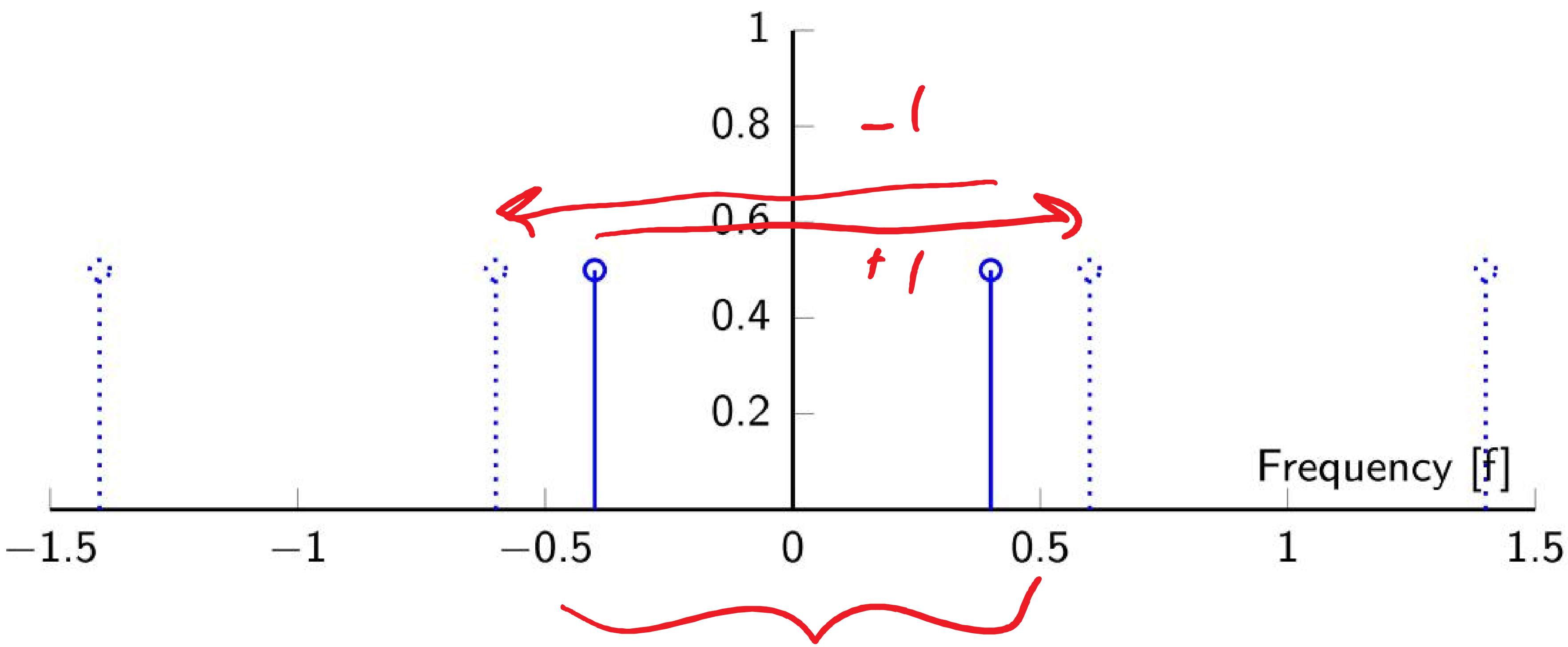
where $F_0 = 400$ and $F_s = 1000$ gives

$$x(n) = \cos\left(2\pi \cdot \frac{400}{1000} \cdot n\right) = \cos(2\pi 0.4n) = \cos(2\pi(0.4 + k)n) \qquad k \text{ integer}$$

The spectrum $|\overset{X}{\cancel{H}}(F)|$ before sampling:



The spectrum $|\cancel{H}(f)|$ after sampling:



One period

The digital frequencies are

$$f_0 = \pm 0.4 \pm k$$

Important:

$$\boxed{\text{The spectrum of a digital signal is periodic with the period } f = 1 \text{ or } \omega = 2\pi.}$$

Example of folding distortion

Consider a signal with two cosine terms.

$$x(t) = \cos(2\pi 400t) + 0.5\cos(2\pi 800t)$$

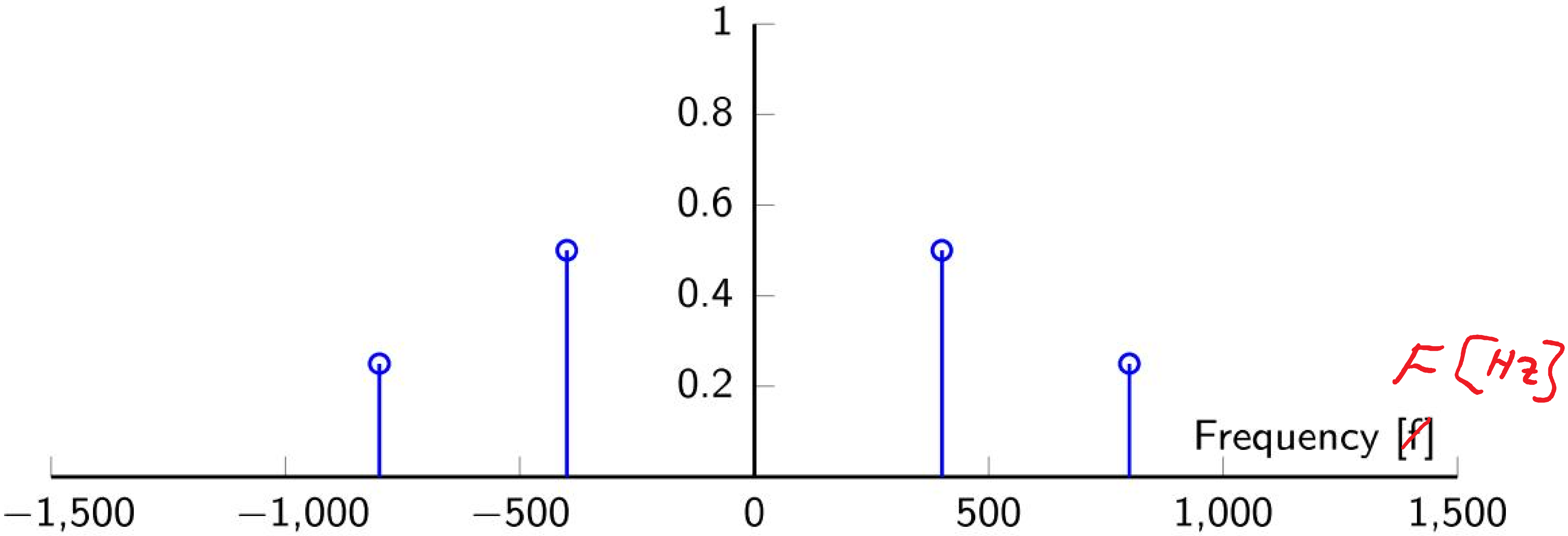
where $F_s = 1000$

$$x(n) = \cos(2\pi 0.4n) + 0.5\cos(2\pi 0.8n)$$

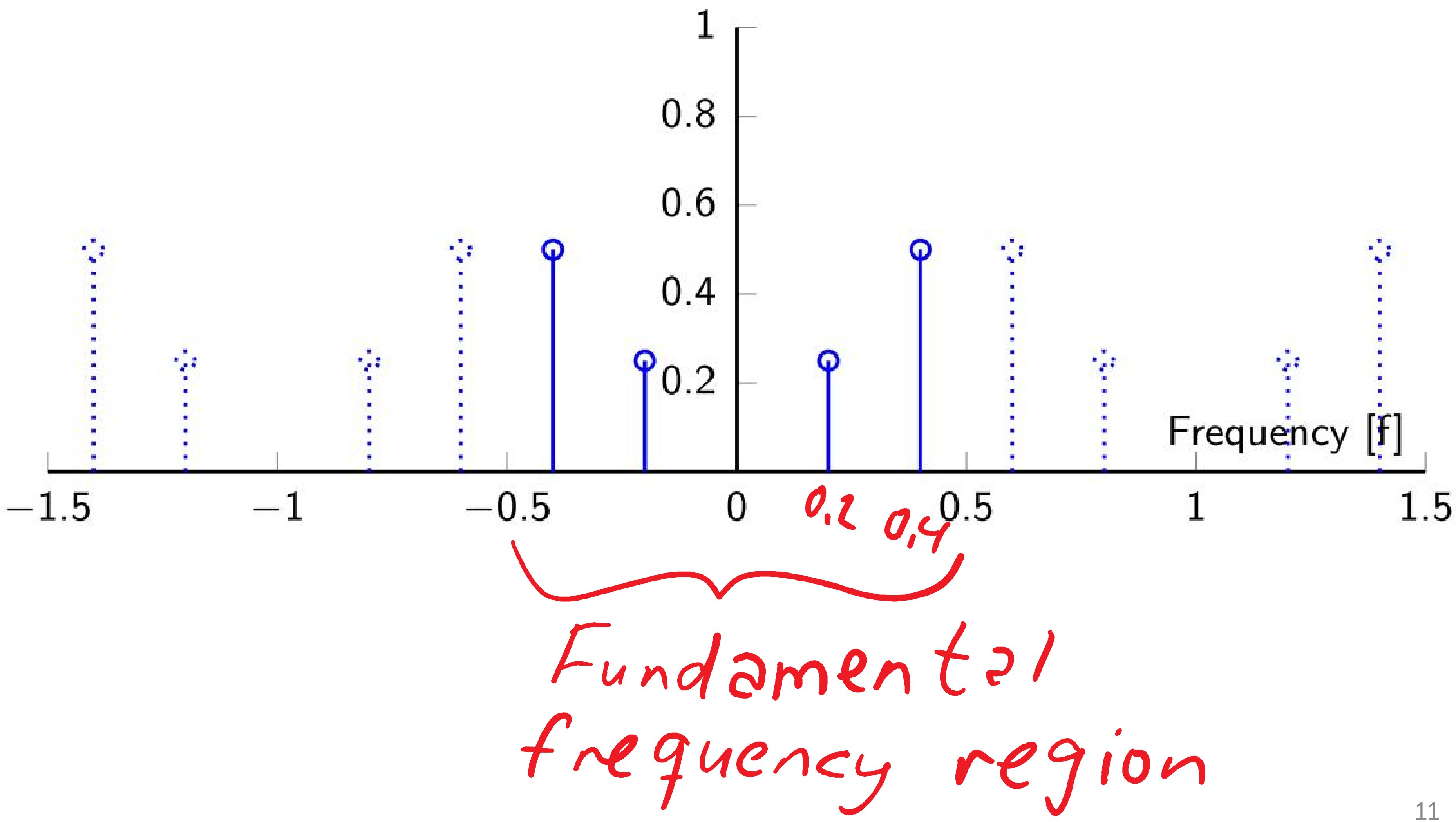
aliasing

$$= \cos(2\pi 0.4n) + 0.5\cos(2\pi \underbrace{(-0.2)}_{0,8-1}n)$$

The spectrum before sampling



The spectrum after sampling:



We will avoid aliasing, i.e. any “false” components if we fulfill the sampling theorem, i.e.

$$\boxed{F_s > 2 \cdot F_{max}}$$

Spectrum for a sampled signal

$\boxed{Here\ F_s > 2 \cdot F_{max}}$

The Fourier transform is defined as

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi F t} \, dt$$

Let $x(n) = x(t)$ where $t = n/F_s$ and change the integral for a sum

$$X(F) \approx \underbrace{\sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi \cdot \frac{F}{F_s} \cdot n}}_{X(f)} \cdot \Delta t = X(f) \cdot \frac{1}{F_s}$$

$\left\{ \text{Since } \Delta t = T = \frac{1}{F_s} \right\}$

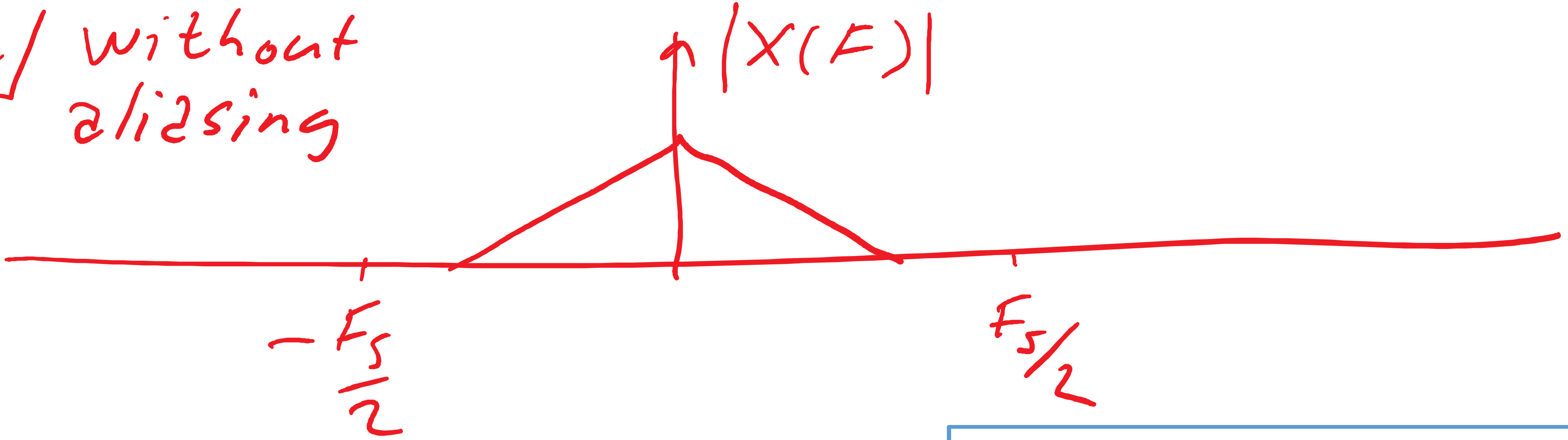
$\Longleftrightarrow$ $\boxed{X(f) = X(F) \cdot F_s}$

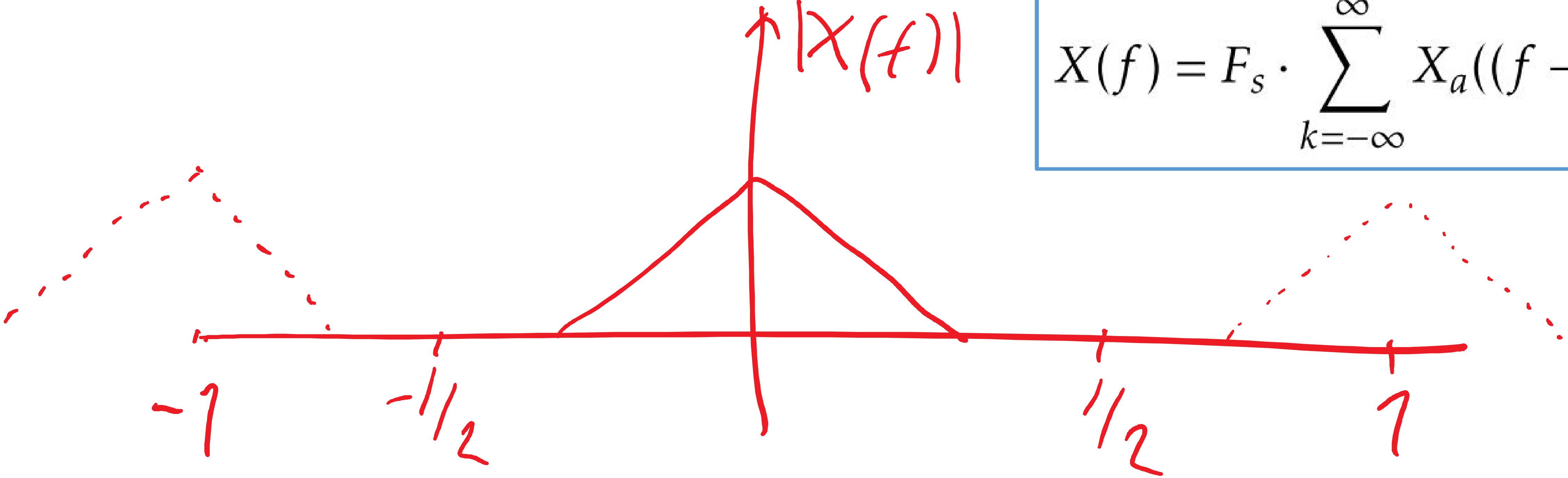
In general (even if we do not fulfill the sampling theorem) we have:

$$X(f) = F_s \cdot [\ldots + X(F - 2F_s) + X(F - F_s) + X(F) + X(F + F_s) + X(F + 2F_s) + \ldots]$$

$$= F_s \cdot \sum_{k=-\infty}^{\infty} X(F - kF_s)$$

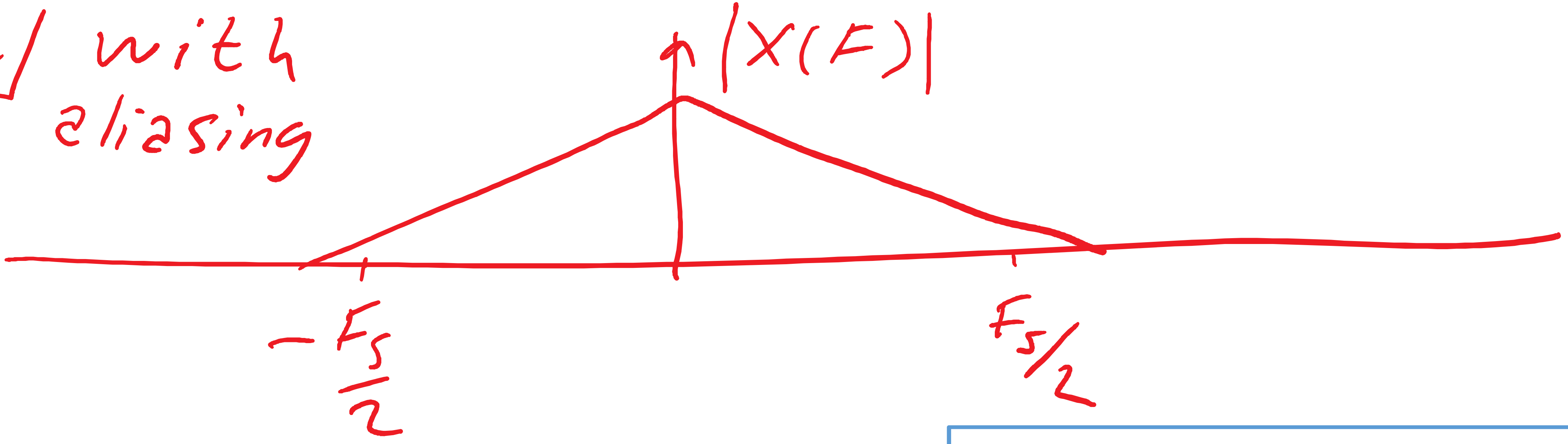
Ex/ without aliasing

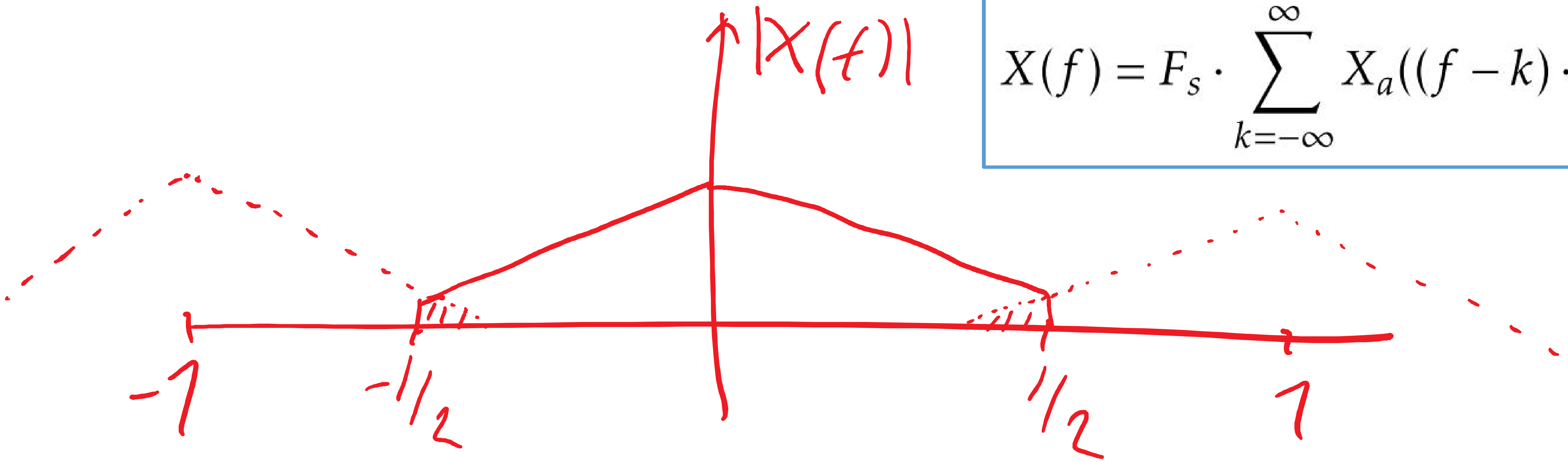




$$X(f) = F_s \cdot \sum_{k=-\infty}^{\infty} X_a((f-k)\cdot F_s)$$

Ex/ with aliasing





$$X(f) = F_s \cdot \sum_{k=-\infty}^{\infty} X_a((f-k) \cdot F_s)$$

Proof

Use the definitions of continuous time Fourier transform and discrete time Fourier transform, and it’s inverses

$$X_a(F) = \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi F t}\, dt \quad \Leftrightarrow \quad x_a(t) = \int_{-\infty}^{\infty} X_a(\overset{F}{\cancel{t}}) e^{j2\pi F t}\, dF$$

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n} \quad \Leftrightarrow \quad x(\overset{n}{\cancel{t}}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f) e^{j2\pi f \overset{n}{\cancel{t}}}\, df$$

Periodic sampling imposes a relation between t and n of

$$t = nT = \frac{n}{F_s}$$

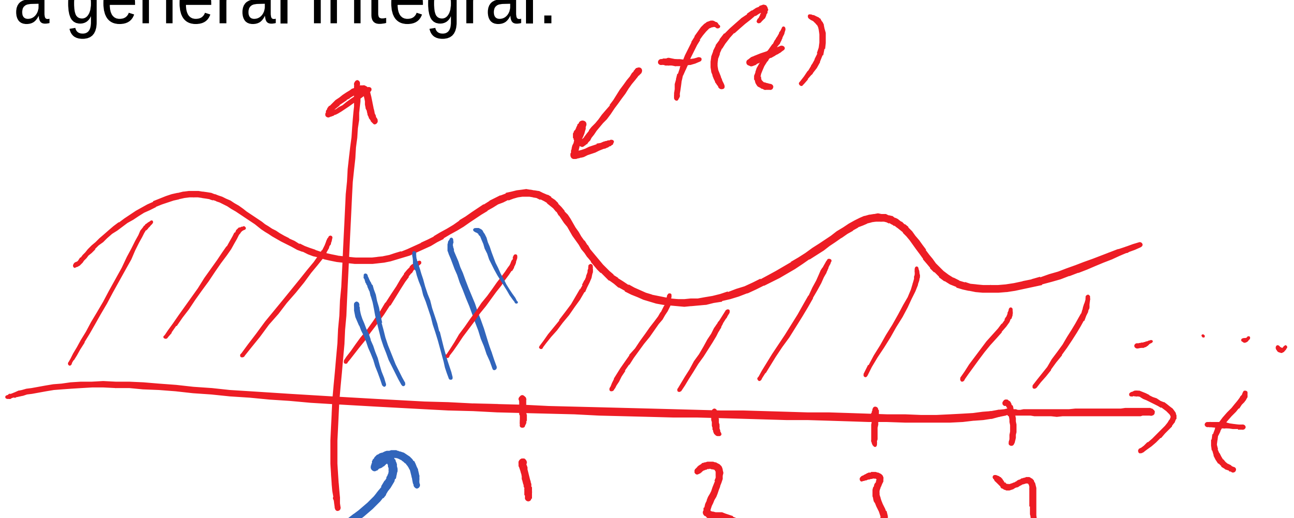
If $x(n) = x_a(nT)$ then

$$x(n) = \underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} X(f)\mathrm{e}^{\mathrm{j}2\pi f n}\,\mathrm{d}f}_{x(n)} = \underbrace{\int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{\mathrm{j}2\pi \cdot \frac{F}{F_s}\cdot n}\,\mathrm{d}F}_{x_a(n \cdot T)}$$

t

Before we continue we look at a general integral:

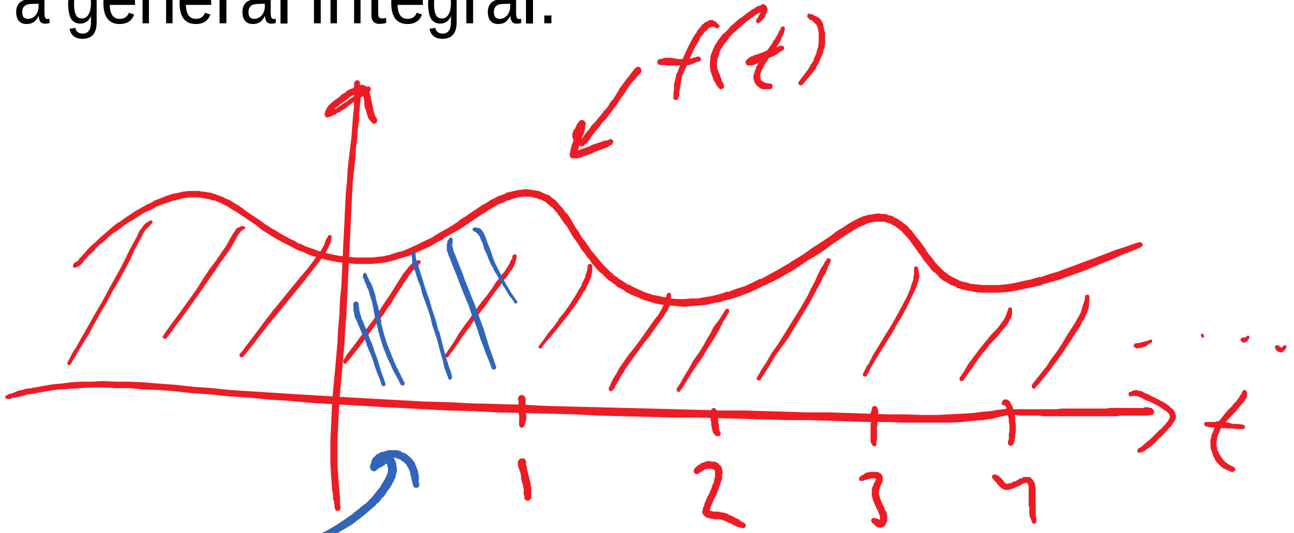
$$\int_{-\infty}^{\infty} f(t)\,dt \Rightarrow$$



$$\Leftrightarrow \dots + \int_{-1}^{0} f(t)\,dt + \int_{0}^{1} f(t)\,dt + \int_{1}^{2} f(t)\,dt + \dots$$

Before we continue we look at a general integral:

$$\int_{-\infty}^{\infty} f(t)\,dt \Rightarrow$$



$$\Leftrightarrow \dots + \int_{-1}^{0} f(t)\,dt + \int_{0}^{1} f(t)\,dt + \int_{1}^{2} f(t)\,dt + \dots$$

$$\Leftrightarrow \dots + \int_{0}^{1} f(t-1)\,dt + \int_{0}^{1} f(t)\,dt + \int_{0}^{1} f(t+1)\,dt + \dots$$

$$x(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f)e^{j2\pi f n}\,df$$

$$= \int_{-\infty}^{\infty} X_a(F)e^{j2\pi\cdot\frac{F}{F_s}\cdot n}\,dF$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[F_s \cdot \sum_{k=-\infty}^{\infty} X_a((f-k)\cdot F_s)\right] \cdot e^{j2\pi f n}\, df$$

We have now shown:

$$X(f) = F_s \cdot \sum_{k=-\infty}^{\infty} X_a((f-k)\cdot F_s)$$

Example with folding distortion and phase addition

Assume a signal with a cosine and a sine term.

$$x(t) = \cos(2\pi 400 t) + \sin(2\pi 600 t)$$

$$= \frac{1}{2} \cdot e^{j2\pi 400t} + \frac{1}{2} \cdot e^{-j2\pi 400t} + \frac{1}{j2} \cdot e^{j2\pi 600t} - \frac{1}{j2} \cdot e^{-j2\pi 600t}$$

Sampling with $F_s = 1000$ yields

$$x(n) = \cos(2\pi 0.4 n) + \sin(2\pi 0.6 n)$$

$$x(n) = \cos(2\pi 0.4n) + \sin(2\pi 0.6n)$$

$$= \frac{1}{2} \cdot e^{j2\pi 0.4n} + \frac{1}{2} \cdot e^{-j2\pi 0.4n} + \frac{1}{j2} \cdot e^{j2\pi 0.6\cancel{t}\,n} - \frac{1}{j2} \cdot e^{-j2\pi 0.6\cancel{t}\,n}$$

$$\left[e^{j2\pi 0.6n} = e^{-j2\pi 0.4n} \quad \text{and} \quad \frac{1}{j2} = \frac{1}{2} \cdot e^{-j\pi/2} \right]$$

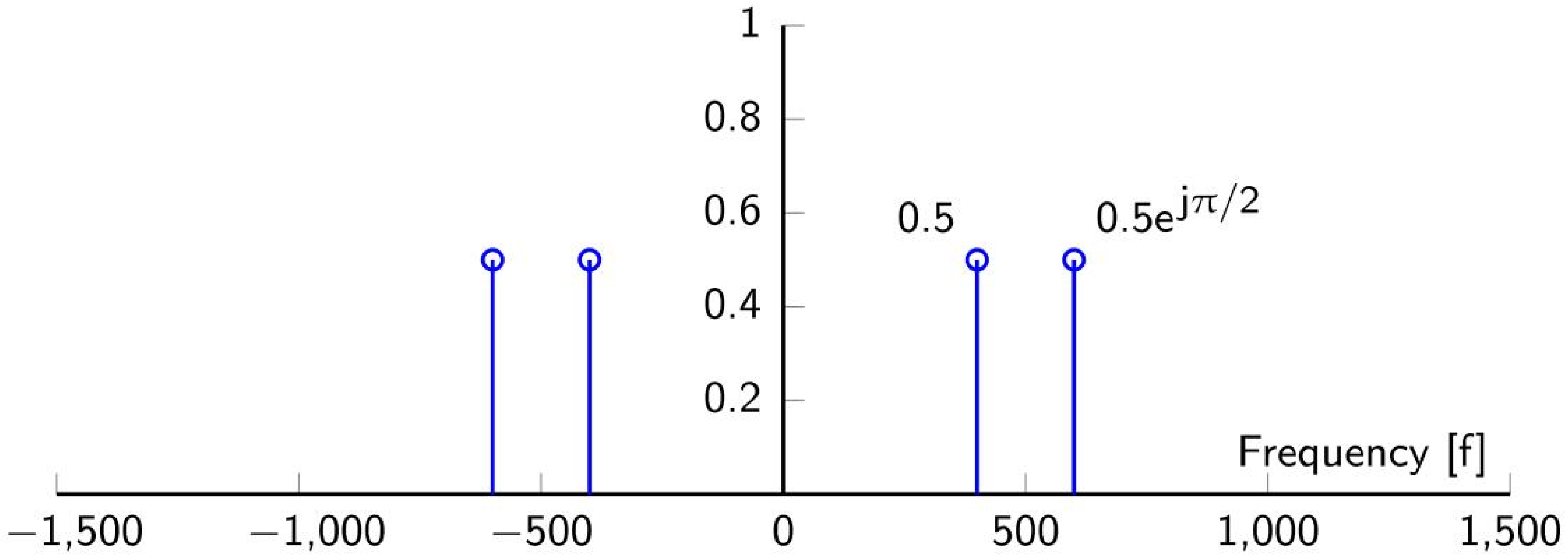
$$= \frac{1}{2} \cdot e^{j2\pi 0.4n} + \frac{1}{2} \cdot e^{-j2\pi 0.4n} - \frac{1}{2} \cdot e^{-j\pi/2} e^{j2\pi 0.4\cancel{t}\,n} + \frac{1}{2} \cdot e^{-j\pi/2} e^{-j2\pi 0.4\cancel{t}\,n}$$

$$= \frac{1}{2} \cdot \left(1 + e^{j\cdot\frac{1}{2}\cdot\pi}\right) \cdot e^{j2\pi 0.4n} + \frac{1}{2} \cdot \left(1 + e^{-j\cdot\frac{1}{2}\cdot\pi}\right) \cdot e^{-j2\pi 0.4n}$$

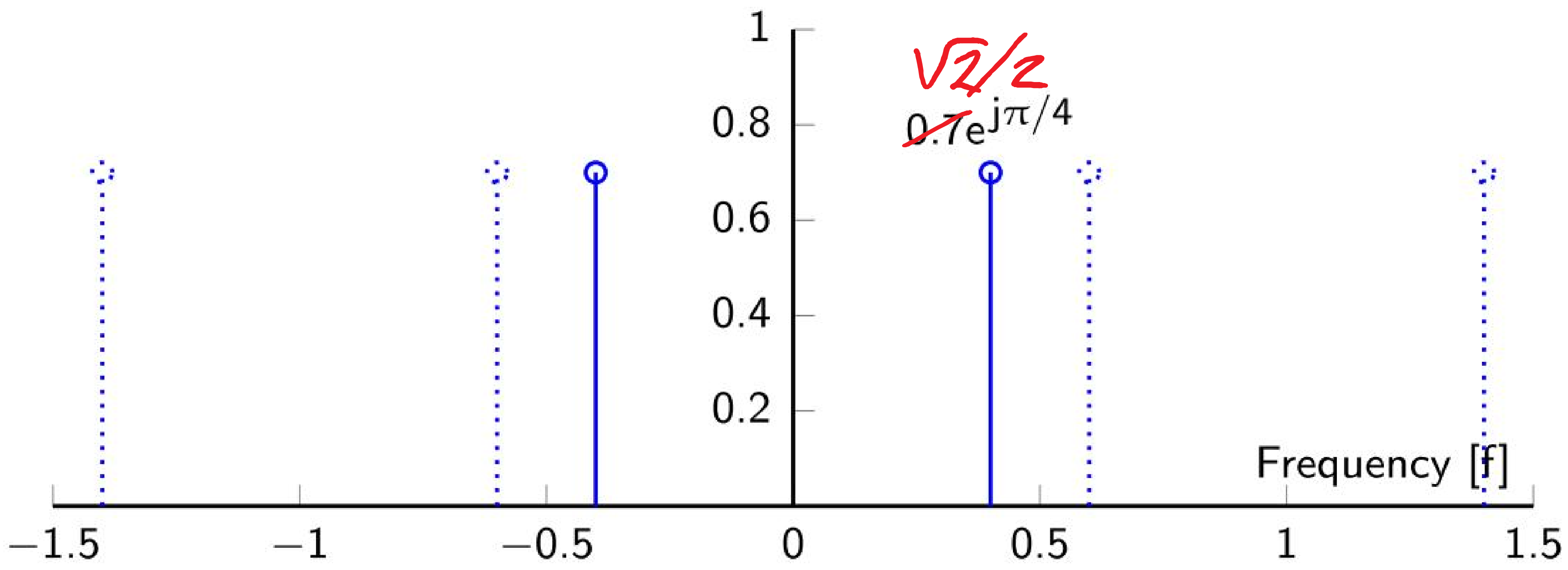
$$= \frac{\sqrt{2}}{2} \cdot e^{j2\pi 0.4n + \frac{\pi}{4}} + \frac{\sqrt{2}}{2} \cdot e^{-j2\pi 0.4n - \frac{\pi}{4}}$$

$$= \sqrt{2} \cdot \cos\left(2\pi 0.4n + \frac{\pi}{4}\right)$$

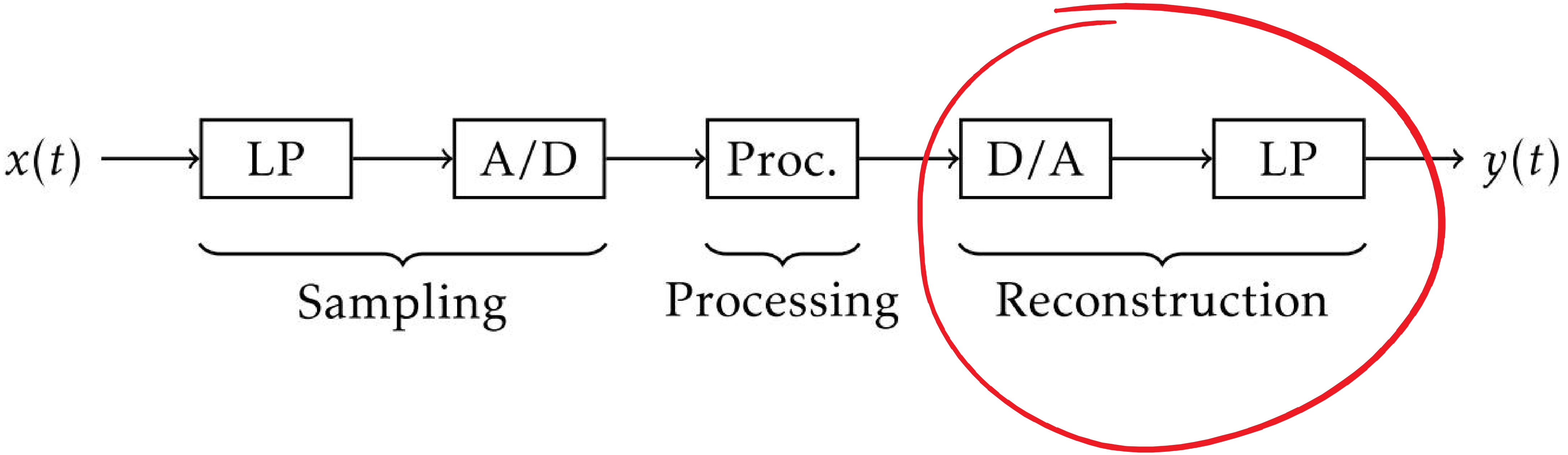
Spectrum before sampling (the analog signal):



Spectrum after sampling (the discrete signal):



How do we do reconstruction and D/A-conversion? (page 387–388, 395–397)



We select the part of the spectrum ~~ni~~ in the interval

$$-0.5 < f < 0.5$$

equivalent to

$$-\frac{F_s}{2} < F < \frac{F_s}{2} \qquad \text{i Hz}$$

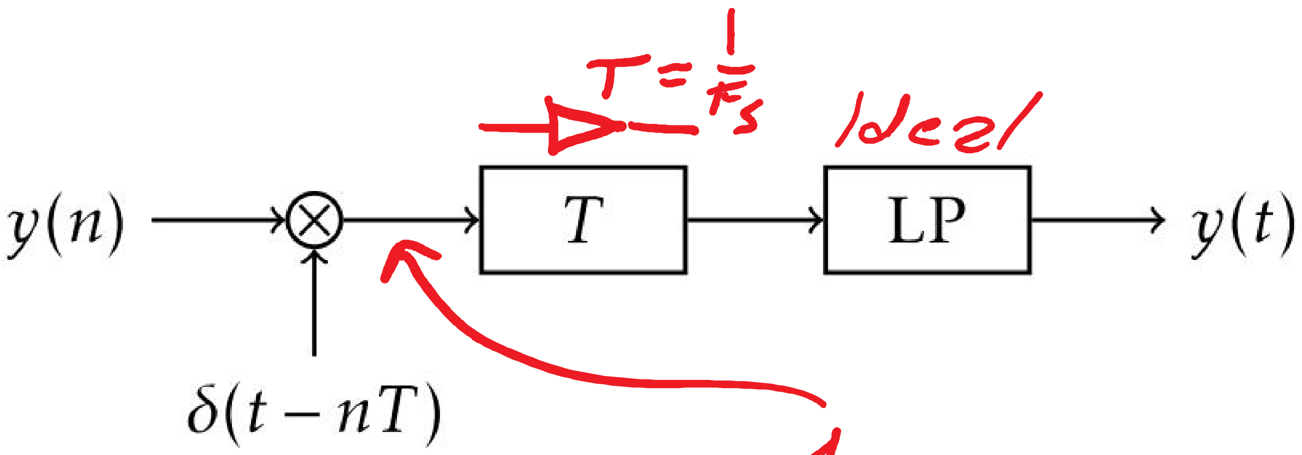
using a low pass filter.

$$y(n) \longrightarrow \underbrace{\boxed{\text{D/A}}}_{\text{let us look at this}} \longrightarrow \boxed{\text{LP}} \longrightarrow y(t)$$

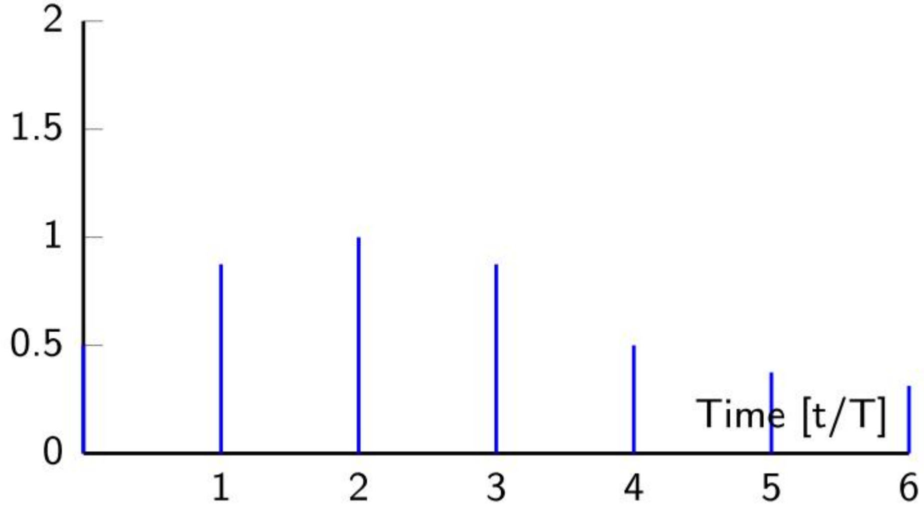
Ideal reconstruction

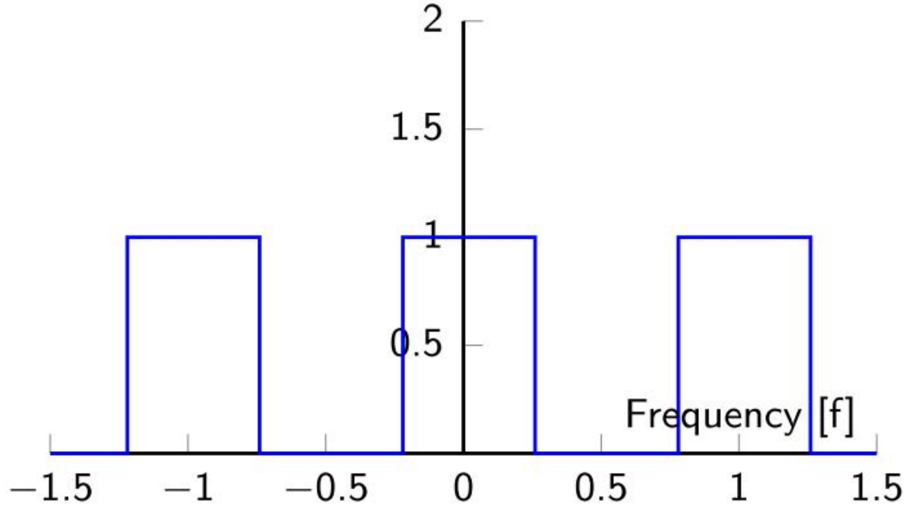
recall, we had

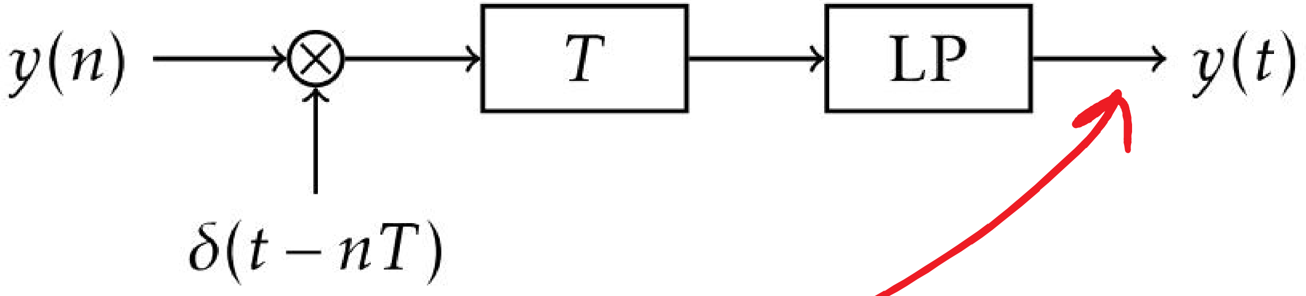
$$\boxed{X(f) = X(F) \cdot F_s}$$



Analog impulse signal of the discrete signal:



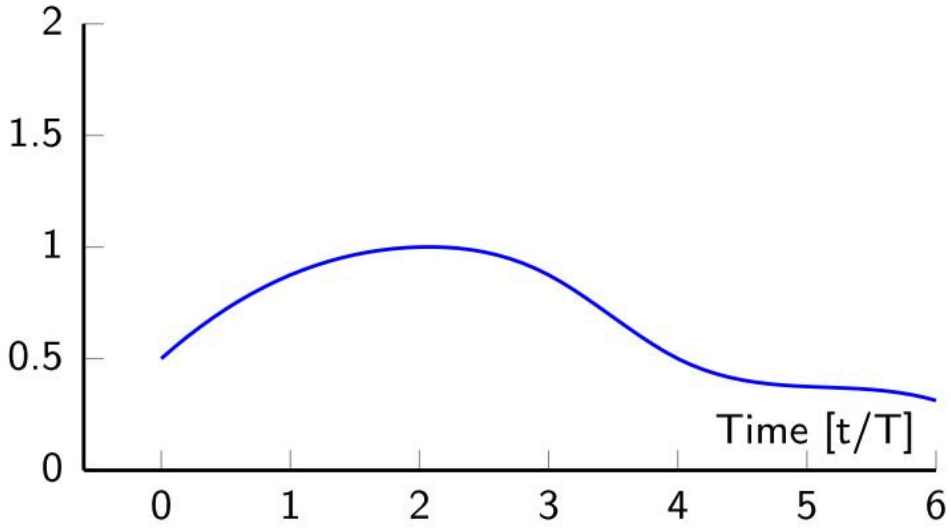


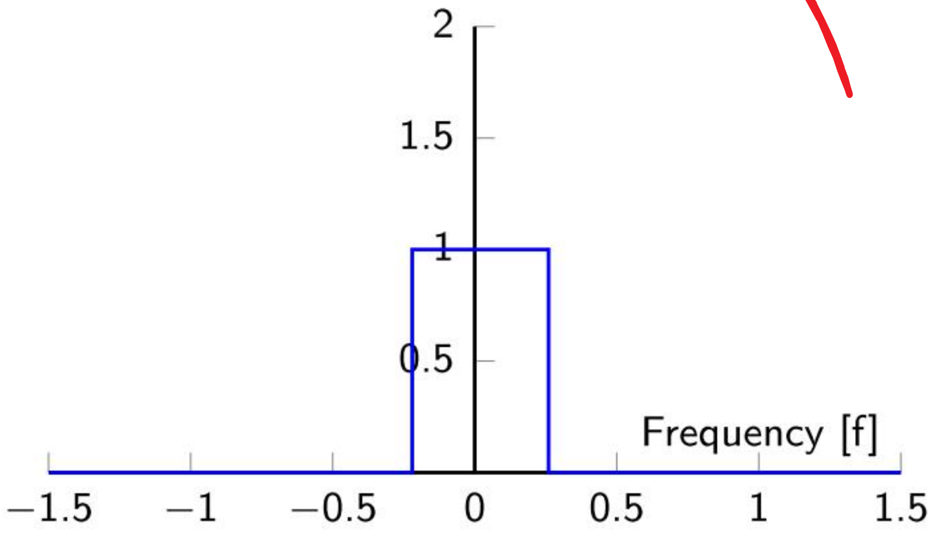


Y(ω)

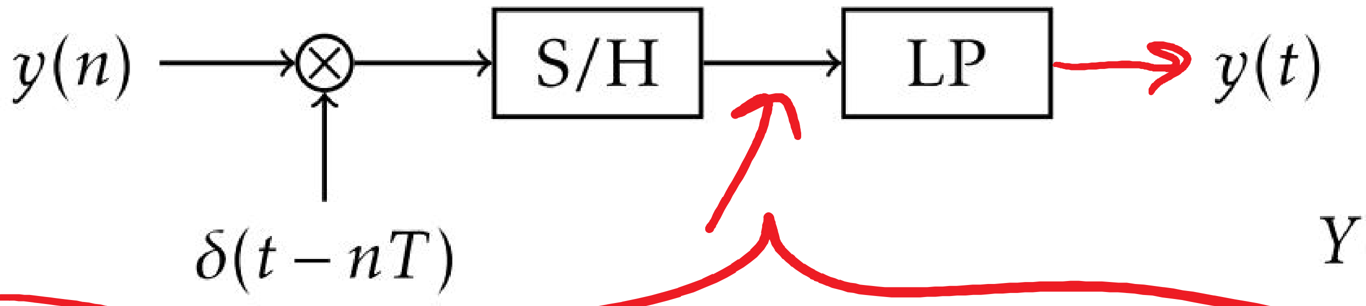
$$Y(F) = \frac{1}{F_s} \cdot \cancel{X(\omega)} \cdot H_{\mathrm{LP}}(\omega)$$

Low pass filtered signal:





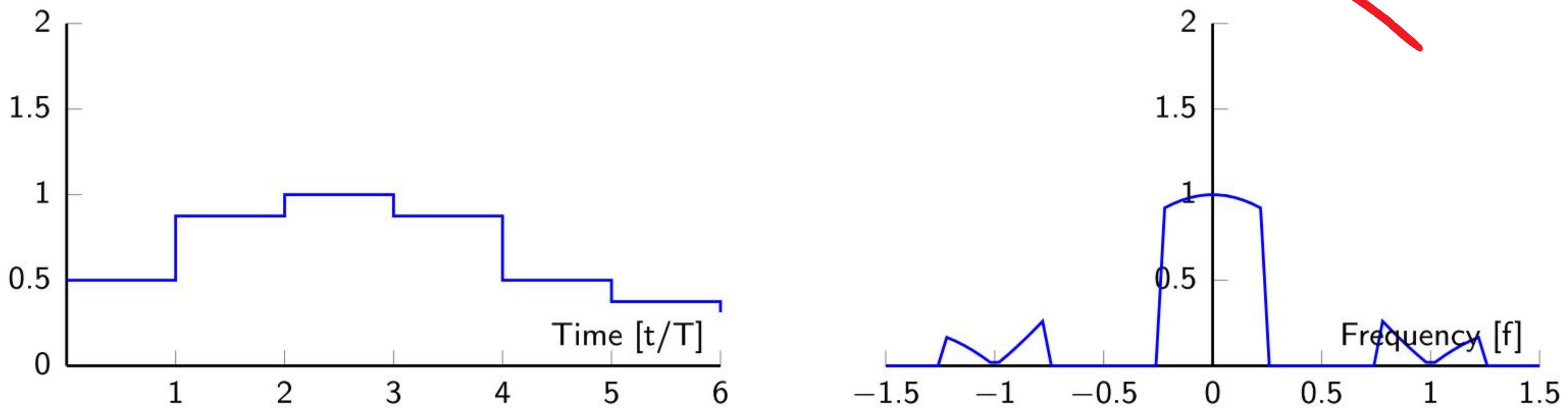
Reconstruction with sample-and-hold

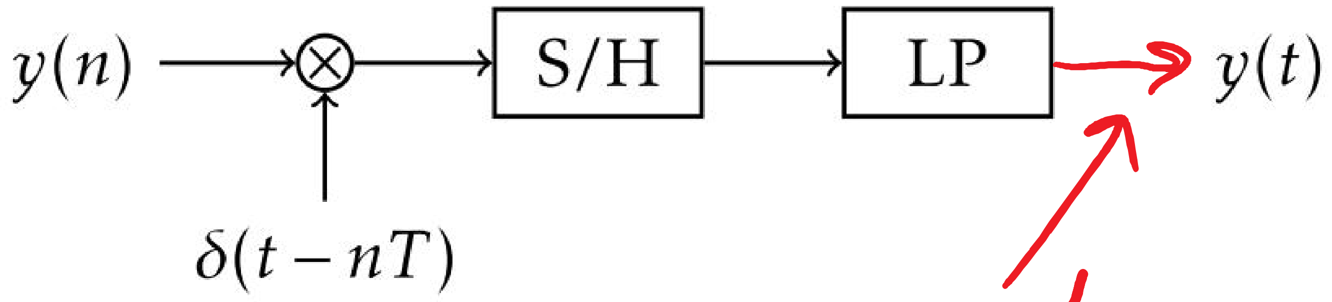


$Y(\omega)$

$$Y(F) = \cancel{X(\omega)} \cdot H_{\mathrm{SH}}(F) \cdot H_{\mathrm{LP}}(F)$$

Analog sample-and-hold signal of the discrete signal:

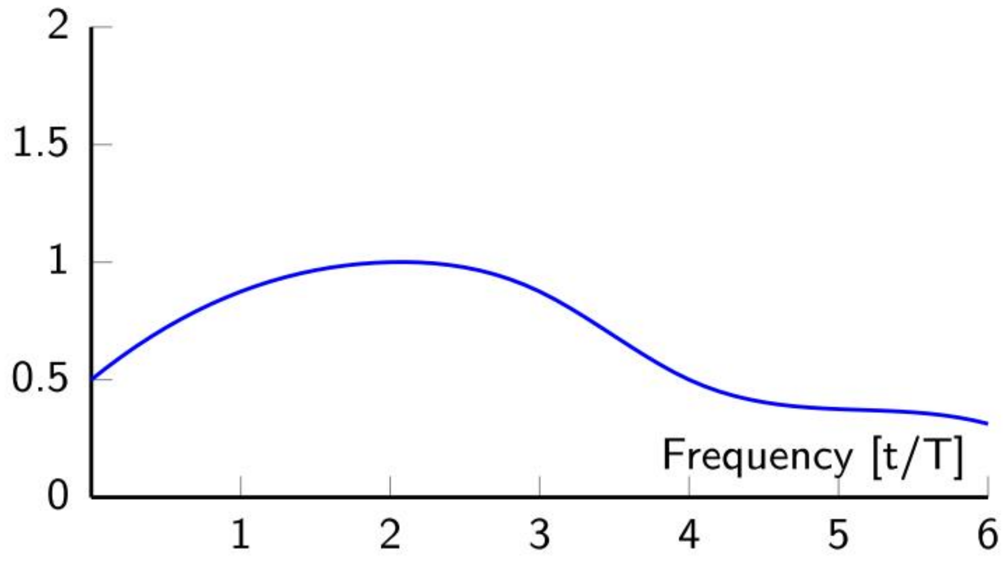


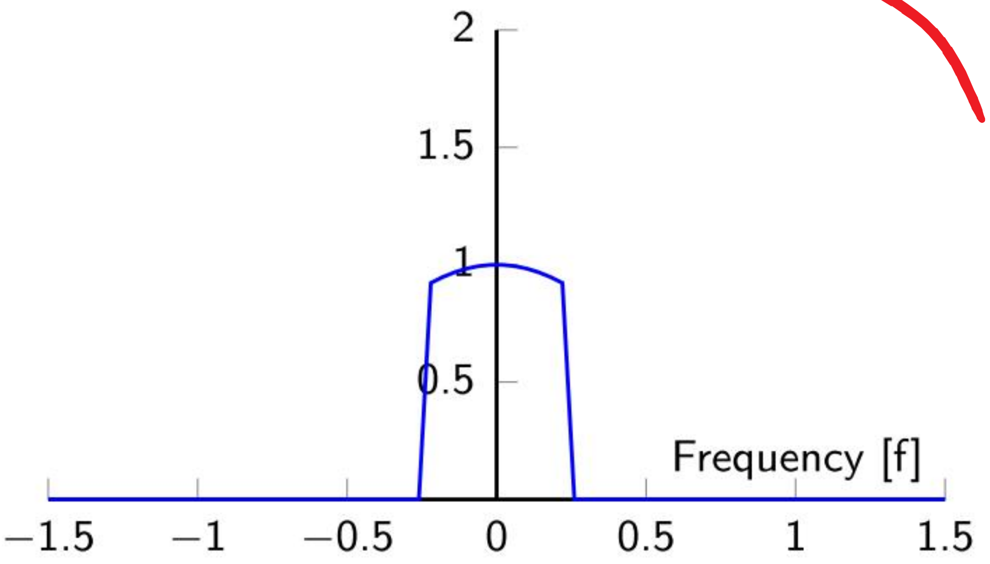


Y(ω)

$$Y(F) = \cancel{X(\omega)} \cdot H_{\mathrm{SH}}(F) \cdot H_{\mathrm{LP}}(F)$$

Low pass filtered signal:





Interpolation (increasing the sample rate)

symbol

$$F_s \longrightarrow \boxed{\uparrow I} \longrightarrow \quad F_s' = F_s \cdot I$$

Decimation (reducing the sample rate)

symbol

$$F_s \longrightarrow \boxed{\downarrow D} \longrightarrow \quad F_s' = \frac{F_s}{D}$$

Ex: Interpolation

Interpolation (increasing the sample rate)

Given:

$$x(n) = \sin(2\pi f_0 n) = \left\{ \ldots \quad x(-1) \quad x(0) \quad x(1) \quad x(2) \quad \ldots \right\} \quad \text{where } f_0 = 0.4 \tag{39}$$

Find: $Y(f)$ of the sequence

$$y(n) = \left\{ \ldots \quad x(-1) \quad 0 \quad x(0) \quad 0 \quad x(1) \quad 0 \quad x(2) \quad \ldots \right\} \tag{40}$$

Solution:

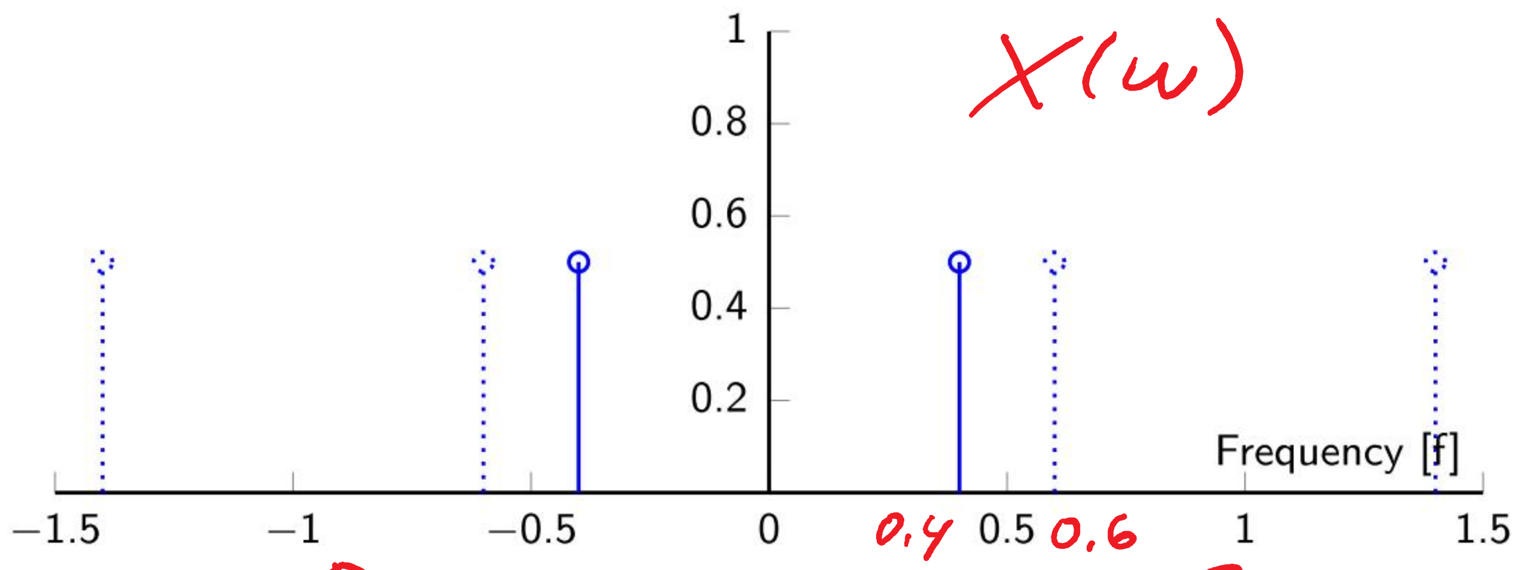
$$y(n) = \begin{cases} x(n/2) & \text{for } n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

$$Y(\omega) = \sum_{n} y(n) e^{-j\omega n} \qquad [\text{let } n' = n/2 \text{ and } n = 2n']$$

$$= \sum_{n'} x(n') e^{-j\omega 2n'} = X(2\omega)$$

One period of $Y(\omega)$ corresponds to 2 periods of $X(\omega)$

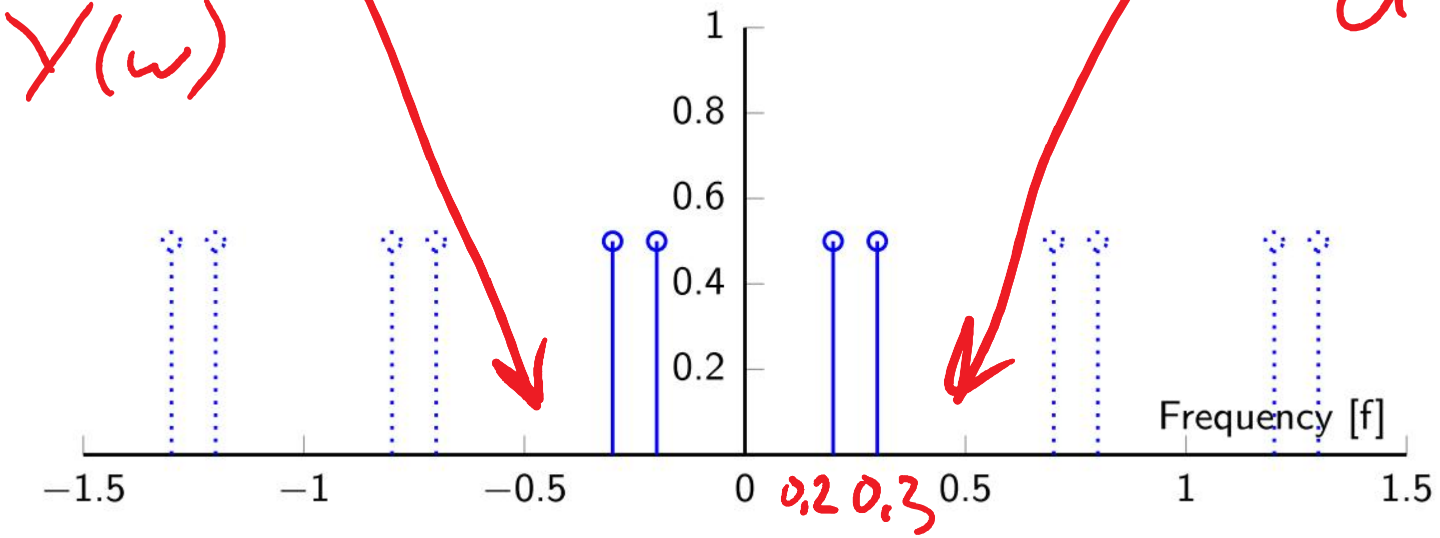
Before interpolation:



After interpolation:

Y(w)

divide by 2



Therefore, we get

$$y(n) = \sin(2\pi 0.2n) + \sin(2\pi 0.3n)$$

A general formula for the resulting frequencies is given by;

Frequencies after interpolation are

$$\boxed{f' = \frac{\pm\frac{F}{F_s} \pm k}{I}}$$

$$-0.5 < f' < 0.5$$

We had

$f_0 = \pm 0.4 \pm k =$

$\pm 0.4 \pm 0.6 \pm 1.4 \pm 1.6$

$\Rightarrow f' = \pm 0.2 \pm 0.3$?

Ex: Decimation

Decimation (reducing the sample rate)

Given:

$$x(n) = \sin(2\pi f_0 n)$$

$$\underbrace{\text{Assume a sinusoid of 100 Hz is sampled with a frequency of 8000 Hz.}}$$

$\Rightarrow f_0 = \frac{1}{80}$

$\Rightarrow x(n) = \sin\left(2\pi \cdot \frac{1}{80} \cdot n\right)$

Find: $Y(\omega)$ of the sequence

$$y(n) = \cancel{X}^{x}(Dn) \qquad \text{for } D = 4$$

The signal after decimation by $D = 4$ is

$$y(n) = x(Dn) = \sin\left(2\pi 4 \cdot \frac{1}{80} \cdot n\right) = \sin\left(2\pi \cdot \frac{1}{20} \cdot n\right)$$

A general formula for the resulting frequencies is given by;

Frequencies after decimation are

$$f' = \pm\frac{F}{F_s} \cdot D \pm k$$

Example in Matlab;

```
N=2048;
x=cos(2*pi*0.4*(0:N-1));
figure,plot(linspace(-0.5,0.5-1/N,N),abs(fftshift(fft(x,N)))/N)

y=kron(x,[1 0]); %Interpolation by factor 2
figure,plot(linspace(-0.5,0.5-1/N,N),abs(fftshift(fft(y,N)))/N)

x=cos(2*pi*1/80*(0:N-1));
y=x(1:4:end); %Decimation by factor 4
figure,plot(linspace(-0.5,0.5-1/N,N),abs(fftshift(fft(y,N)))/N)
```

Ex: **Decimation with folding**

$F_s = 8000\,Hz$

$$\frac{3200}{8000} = 0.4 = \frac{2}{5}$$

Given: Previous example with $f_0 = 3200\,\text{Hz}$ and $D = 3$.

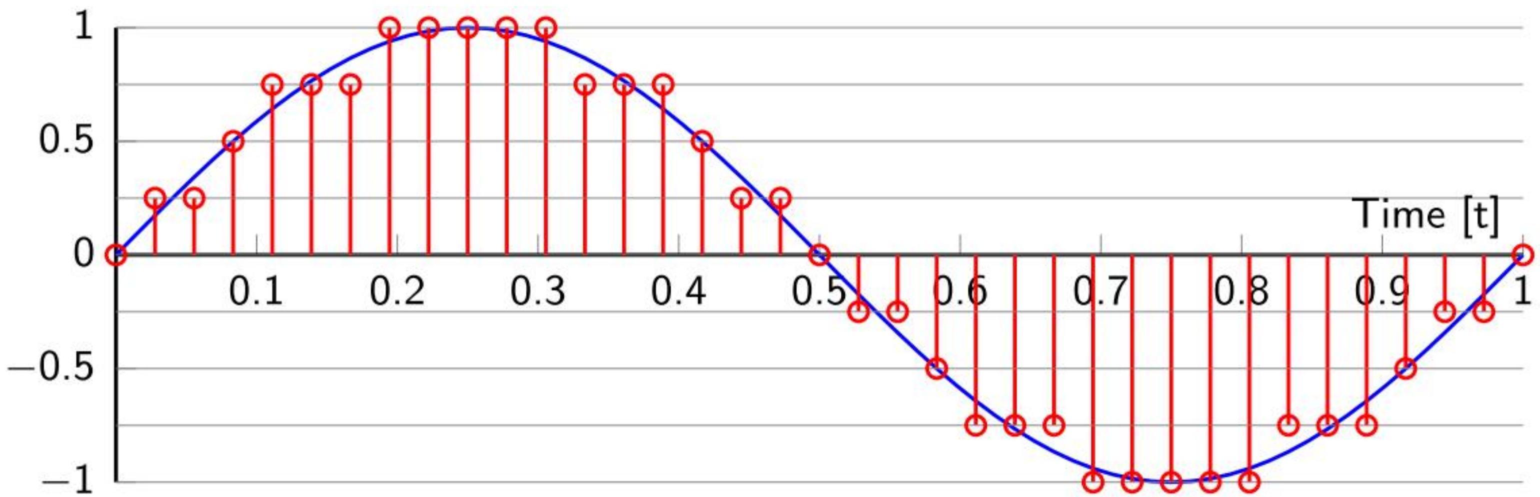
$$y(n) = x(Dn) = \sin\left(2\pi 3 \cdot \frac{2}{5} \cdot n\right) = \sin\left(2\pi \cdot \frac{6}{5} \cdot n\right) = \sin\left(2\pi \cdot \frac{1}{5} \cdot n\right)$$

$$= 1 + \frac{1}{5}$$

Quantization errors during A/D-conversion (page 403–408) Read summarily

The quantization effect:

$$x(t) = A \cdot \sin(\omega_0 t)$$



Signal to Quantization noise ratio

$$\text{SQNR} = 10 \log_{10} \frac{P_s}{P_q} = 1.76 + 2b \approx \boxed{6 \times \text{number of bits}}$$ [dB]